

INHOMOGENOUS RANDOM ZERO SETS.

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ABSTRACT. We construct random point processes in \mathbb{C} that are asymptotically close to a given doubling measure. The processes we construct are the zero sets of random entire functions that are constructed through generalised Fock spaces. We offer two alternative constructions, one via bases for these spaces and another via frames, and we show that for both constructions the average distribution of the zero set is close to the given doubling measure, and that the variance is much less than the variance of the corresponding Poisson point process. We prove some asymptotic large deviation estimates for these processes, which in particular allow us to estimate the ‘hole probability’, the probability that there are no zeroes in a given open bounded subset of the plane. We also show that the ‘smooth linear statistics’ are asymptotically normal, under an additional regularity hypothesis on the measure. These generalise previous results by Sodin and Tsirelson for the Lebesgue measure.

1. INTRODUCTION

In this paper we are interested in random point processes that mimic a given σ -finite measure μ on the complex plane. The classical example is the inhomogenous Poisson point process, which we consider in the following manner. Fix a parameter $L > 0$ and let N_L be the Poisson random measure on \mathbb{C} with intensity $L\mu$, that is,

- N_L is a random measure on \mathbb{C} ,
- For every measurable $A \subset \mathbb{C}$, $N_L(A)$ is a Poisson random variable with mean $L\mu(A)$, and
- If A and B are disjoint then $N_L(A)$ and $N_L(B)$ are independent.

Such an N_L always exists, see for example [Sat99, Proposition 19.4]. Suppose that $\psi \in L^1(\mu) \cap L^2(\mu)$ and define

$$N(\psi, L) = \frac{1}{L} \int_{\mathbb{C}} \psi(z) dN_L(z).$$

Then (see [Sat99, Proposition 19.5])

$$(1) \quad \mathbb{E}[N(\psi, L)] = \int_{\mathbb{C}} \psi d\mu$$

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and, writing \mathbb{V} for the variance,

$$(2) \quad \mathbb{V}[N(\psi, L)] = \frac{1}{L} \int_{\mathbb{C}} |\psi|^2 d\mu.$$

In contrast to the Poisson point process, the zero sets of random analytic functions are known to be more ‘rigid’ processes, in particular these processes exhibit ‘local repulsion’ (see [HKPV09, Chapter 1]). We will construct a random zero set such that (1) continues to hold (at least for smooth ψ in the limit $L \rightarrow \infty$, see Theorem 1) but with a variance that decays faster than L^{-2} , in contrast to (2) (Theorem 2). In fact we will also have

$$N(\psi, L) \rightarrow \int_{\mathbb{C}} \psi d\mu \text{ as } L \rightarrow \infty$$

almost surely, as well as being true in mean (Theorem 1).

As a further measure of the ‘rigidity’ of our process we note that the ‘hole probability’ for the Poisson point process is, by definition,

$$\mathbb{P}[N_L(A) = 0] = e^{-L\mu(A)}$$

for any $A \subset \mathbb{C}$ whereas we shall see that the ‘hole probability’ for the zero sets we construct decays at least like e^{-cL^2} for some $c > 0$.

This problem has already been considered when μ is the Lebesgue measure on the plane ([ST04] and [ST05]) and the resultant zero-sets are invariant in distribution under plane isometries. We are interested in generalising this construction to other measures, where we cannot expect any such invariance to hold. We begin by recalling the following definition.

Definition 1. A nonnegative Borel measure μ in \mathbb{C} is called *doubling* if there exists $C > 0$ such that

$$\mu(D(z, 2r)) \leq C\mu(D(z, r))$$

for all $z \in \mathbb{C}$ and $r > 0$. We denote by C_μ the infimum of the constants C for which the inequality holds, which is called the *doubling constant* for μ .

Let μ be a doubling measure and let ϕ be a subharmonic function with $\mu = \Delta\phi$. Canonical examples of such functions are given by $\phi(z) = |z|^\alpha$ where $\alpha > 0$ (the value $\alpha = 2$ corresponds of course to the Lebesgue measure). The function $\phi(z) = (\operatorname{Re} z)^2$ gives a non-radial example, and more generally one can take ϕ to be any subharmonic, non-harmonic, (possibly non-radial) polynomial. We define, for $z \in \mathbb{C}$, $\rho_\mu(z)$ to be the radius such that $\mu(D(z, \rho_\mu(z))) = 1$. We shall normally ignore the dependence on μ and simply write $\rho(z)$.

Consider the generalised Fock space

$$\mathcal{F}_\phi^2 = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}_\phi^2}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)} \frac{dm(z)}{\rho(z)^2} < +\infty\}$$

where m is the Lebesgue measure on the plane. We note that, as in [Chr91], the measure $\frac{dm(z)}{\rho(z)^2}$ can be thought of as a regularisation of the measure $d\mu(z)$. The classical Bargmann-Fock space corresponds to $\phi(z) = |z|^2$. Let $(e_n)_n$ be an orthonormal basis for the space \mathcal{F}_ϕ^2 and $(a_n)_n$ be

a sequence of independent standard complex Gaussian random variables (that is, the probability density of each a_n is $\frac{1}{\pi} \exp(-|z|^2)$ with respect to the Lebesgue measure on the plane; we denote this distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$). Consider the Gaussian analytic function (GAF) defined by

$$g(z) = \sum_n a_n e_n(z).$$

This sum almost surely defines an entire function (see for example [HKPV09, Lemma 2.2.3]). The covariance kernel associated to this function is given by (note that $\mathbb{E}[g(z)] = 0$ for all $z \in \mathbb{C}$)

$$\mathcal{K}(z, w) = \mathbb{E}[g(z)\overline{g(w)}] = \sum_n e_n(z)\overline{e_n(w)}$$

which is the reproducing kernel for the space \mathcal{F}_{ϕ}^2 . Moreover the distribution of the random analytic function g is determined by the kernel \mathcal{K} so it does not matter which basis we chose.

We are interested in studying the zero set $\mathcal{Z}(g)$, and a first observation is that since $g(z)$ is a mean-zero, normal random variable with variance $\mathcal{K}(z, z) \neq 0$ (see Proposition 14), g has no deterministic zeroes. Furthermore the random zeroes of g are almost surely simple ([HKPV09, Lemma 2.4.1]). We study the zero set $\mathcal{Z}(g)$ through the counting measure

$$n_g = \frac{1}{2\pi} \Delta \log |g|$$

(this equality is to be understood in the distributional sense). The Edelman-Kostlan formula ([Sod00, Theorem 1] or [HKPV09, Section 2.4]) for the density of zeroes gives

$$\mathbb{E}[n_g(z)] = \frac{1}{4\pi} \Delta \log \mathcal{K}(z, z) dm(z).$$

We finally note that $\Delta \log \mathcal{K}(z, z) \simeq \frac{1}{\rho(z)^2}$ (see Section 2.2) which, as we have already noted, can be viewed as a regularisation of the measure μ .

We will modify this construction by re-scaling the weight ϕ , so that the zeroes will be even better distributed. Specifically, let L be a positive parameter and consider instead the weight $\phi_L = L\phi$ (and $\rho_L = \rho_{L\mu}$). For each L we take a basis $(e_n^L)_n$ for the space $\mathcal{F}_L^2 = \mathcal{F}_{\phi_L}^2$ and define

$$(3) \quad g_L(z) = \sum_n a_n e_n^L(z)$$

and

$$\mathcal{K}_L(z, w) = \mathbb{E}[g_L(z)\overline{g_L(w)}] = \sum_n e_n^L(z)\overline{e_n^L(w)}.$$

The following result states that the corresponding zero set, suitably scaled, is well distributed with respect to the measure μ for large values of L .

Theorem 1. *Let ψ be a smooth real-valued function with compact support in \mathbb{C} (which we always assume is not identically zero), let n_L be the counting measure on the zero set of g_L and define the random variable $n(\psi, L) = \frac{1}{L} \int \psi dn_L$.*

(a)

$$\left| \mathbb{E}[n(\psi, L)] - \frac{1}{2\pi} \int \psi d\mu \right| \lesssim \frac{1}{L} \int_{\mathbb{C}} |\Delta\psi(z)| dm(z),$$

where the implicit constant depends only on the doubling constant of the measure μ .

(b) If we restrict L to taking integer values then, almost surely,

$$n(\psi, L) \rightarrow \frac{1}{2\pi} \int \psi d\mu$$

as $L \rightarrow \infty$.

The proof of part (b) of this result uses an estimate on the decay of the variance of $n(\psi, L)$ which is interesting by itself.

Theorem 2. For any smooth function ψ with compact support in \mathbb{C}

$$\mathbb{V}[n(\psi, L)] \simeq \frac{1}{L^2} \int_{\mathbb{C}} \Delta(\psi(z))^2 \rho_L(z)^2 dm(z).$$

Remark. We may estimate the dependence on L using (7) to see that the integral decays polynomially in L . If the measure μ is locally flat (see Definition 3) then we see that the variance decays as L^{-3} , just as in [ST04].

In the special case $\phi(z) = |z|^2/2$ (the factor $1/2$ is simply a convenient normalisation) it is easy to see that the set $(\frac{1}{\pi\sqrt{2}} \frac{(\sqrt{L}z)^n}{\sqrt{n!}})_{n=0}^{\infty}$ is an orthonormal basis for the corresponding Fock space, so that the construction just given corresponds to the GAF studied in [ST04] and [ST05]. More generally if $\phi(z) = |z|^\alpha/2$ and $\alpha > 0$ then the set $(\frac{(L^{1/\alpha}z)^n}{c_{\alpha n}})_{n=0}^{\infty}$ is an orthonormal basis for the corresponding Fock space, for some $c_{\alpha n} \simeq \Gamma(\frac{2}{\alpha}n + 1)^{1/2}$ (and the implicit constants depend on α , see the Appendix).

However, besides these special cases, we have very little information about the behaviour of an orthonormal basis for \mathcal{F}_L^2 . For this reason we also study random functions that are constructed via frames.

Definition 2. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. A sequence $(x_n)_n$ in X is said to be a *frame* if there exist $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2$$

for all $x \in X$.

It can be shown that this implies that there exists a sequence $(\tilde{x}_n)_n$ in X (the canonical dual frame) such that

$$x = \sum_n \langle x, \tilde{x}_n \rangle x_n$$

and

$$\frac{1}{B}\|x\|^2 \leq \sum_n |\langle x, \tilde{x}_n \rangle|^2 \leq \frac{1}{A}\|x\|^2$$

for all $x \in X$. Thus a frame can be thought of as a generalisation of a basis that retains the spanning properties of a basis although the elements of the frame are not, in general, linearly independent. (For a proof of the above facts and a general introduction to frames see, for example, [Chr91, Chapter 5].)

We will consider frames for \mathcal{F}_L^2 consisting of normalised reproducing kernels, $k_\zeta(z) = \frac{\mathcal{K}_L(z, \zeta)}{\mathcal{K}_L(\zeta, \zeta)^{1/2}}$ (we ignore the dependence on L to simplify the notation). We consider frames of the form $(k_\lambda)_{\lambda \in \Lambda_L}$ (where the index set $\Lambda_L \subset \mathbb{C}$ is a sampling sequence, see Section 2.2 for the definition). The advantage of this approach is that we have estimates for the size of the reproducing kernel (Theorem 14), and so we also have estimates for the size of the frame elements. We now define

$$(4) \quad f_L(z) = \sum_{\lambda \in \Lambda_L} a_\lambda k_\lambda(z)$$

where a_λ is a sequence of iid $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables indexed by the sequence Λ_L . The covariance kernel for f_L is given by

$$K_L(z, w) = \mathbb{E}[f_L(z) \overline{f_L(w)}] = \sum_{\lambda \in \Lambda_L} k_\lambda(z) \overline{k_\lambda(w)}$$

which satisfies similar estimates to \mathcal{K}_L (see Proposition 17).

Since the proof of Theorem 1 uses only estimates for the size of the covariance kernel we may state an identical theorem for the GAF defined via frames. However in this case we also have the following stronger result.

Theorem 3. *Let n_L be the counting measure on the zero set of the GAF f_L defined via frames (4), ψ be a smooth real-valued function with compact support in \mathbb{C} , $n(\psi, L) = \frac{1}{L} \int \psi dn_L$.*

(a)

$$\left| \mathbb{E}[n(\psi, L)] - \frac{1}{2\pi} \int \psi d\mu \right| \lesssim \frac{1}{L} \int_{\mathbb{C}} |\Delta \psi(z)| dm(z),$$

where the implicit constant depends only on the doubling constant of the measure μ .

(b) Let $\delta > 0$. There exists $c > 0$ depending only on δ , ψ and μ such that

$$(5) \quad \mathbb{P} \left[\left| \frac{n(\psi, L)}{\frac{1}{2\pi} \int \psi d\mu} - 1 \right| > \delta \right] \leq e^{-cL^2}.$$

as $L \rightarrow \infty$.

The proof of part (a) is identical to the proof of Theorem 1 (a) (using the appropriate estimates for the covariance kernel of f_L). It is also easy to see, by an appeal to the first Borel-Cantelli Lemma, that the large deviations estimate (5) implies that in this case we also have almost sure convergence exactly as stated in Theorem 1 (b). This result has an obvious corollary.

Corollary 4. *Suppose that n_L is the counting measure on the zero set of the GAF f_L defined via frames (4) and U is an open bounded subset of the complex plane.*

(a)

$$\mathbb{E} \left[\frac{1}{L} n_L(U) \right] \rightarrow \frac{1}{2\pi} \mu(U)$$

as $L \rightarrow \infty$.

(b) *Let $\delta > 0$. There exists $c > 0$ depending only on δ , U and μ such that for sufficiently large values of L*

$$\mathbb{P} \left[\left| \frac{\frac{1}{L} n_L(U)}{\frac{1}{2\pi} \mu(U)} - 1 \right| > \delta \right] \leq e^{-cL^2}.$$

Remark. As before, the large deviations estimate combined with the first Borel-Cantelli Lemma implies that $\frac{1}{L} n_L(U) \rightarrow \frac{1}{2\pi} \mu(U)$ almost surely when L is restricted to integer values.

It is well known that the linear statistics are asymptotically normal for the Poisson point process, and we also show that the smooth linear statistics for our zero sets are asymptotically normal, for large values of L , if the measure μ is locally flat. We shall state and prove this result only for the GAF defined via frames (that is (4)) but it is easy to verify that the proof works equally well for the GAF defined via bases (3) since it relies only on estimates for the size of the covariance kernel.

Theorem 5. *Let ψ be a smooth function with compact support in \mathbb{C} let n_L be the counting measure on the zero set of the GAF defined via frames (4), and suppose that measure μ is locally flat (see Definition 3). Define $n(\psi, L) = \frac{1}{L} \int \psi dn_L$ as before. Then the random variable*

$$\frac{n(\psi, L) - \mathbb{E}[n(\psi, L)]}{\mathbb{V}[n(\psi, L)]^{1/2}}$$

converges in distribution to $\mathcal{N}(0, 1)$ as $L \rightarrow \infty$.

We are also interested in the ‘hole probability’, the probability that there are no zeroes in a region of the complex plane. When we take $\phi(z) = |z|^2/2$ then the asymptotic decay of the hole probability for the zero set of the GAF defined via bases was computed in [ST05], and the more precise version

$$\mathbb{P}[n_{g_L}(D(z_0, r)) = 0] = \exp \left\{ -\frac{e^2}{4} L^2 r^4 (1 + o(1)) \right\}$$

as $L \rightarrow \infty$ was obtained in [Nis10, Theorem 1.1] (we note that the author considers $L = 1$ and discs of large radius centred at the origin, however the results are equivalent by re-scaling and translation invariance). If $\phi(z) = |z|^\alpha/2$ and we consider the random function g_L generated by the bases $(\frac{(L^{1/\alpha} z)^n}{c_\alpha n})_{n=0}^\infty$ then we see that

$$\mathbb{P}[n_{g_L}(D(0, r)) = 0] = \mathbb{P}[n_{g_1}(D(0, rL^{1/\alpha})) = 0].$$

We may now use [Nis11, Theorem 1] to see that

$$\mathbb{P}[n_{g_1}(D(0, rL^{1/\alpha})) = 0] = \exp \left\{ -\frac{\alpha e^2}{8} r^{2\alpha} L^2 (1 + o(1)) \right\}$$

as $L \rightarrow \infty$ (we omit the details), however we no longer have translation invariance.

Our first result says that we always have an upper bound of the form e^{-cL^2} , however we have no estimate for the lower bound in general.

Theorem 6. *Suppose that n_L is the counting measure on the zero set of the GAF g_L defined via bases (3). Let U be a bounded open subset of the complex plane. There exists $c > 0$ depending only on U and μ such that for sufficiently large values of L*

$$\mathbb{P}[n_L(U) = 0] \leq e^{-cL^2}.$$

When we work with frames, because we have estimates for the pointwise decay of the reproducing kernel, we can prove much more. In this case we show that we have the same upper bound (with a different constant) and that the upper bound is sharp (up to constants) under additional assumptions on the decay of the kernel \mathcal{K}_L .

Theorem 7. *Suppose that n_L is the counting measure on the zero set of the GAF f_L defined via frames (4). Let U be an open bounded subset of the complex plane.*

(a) *There exist $c, C > 0$ depending on U and μ , and $\tau \geq 2$ depending only on μ , such that for sufficiently large values of L*

$$e^{-CL^\tau} \leq \mathbb{P}[n_L(U) = 0] \leq e^{-cL^2}.$$

(b) *If the reproducing kernel \mathcal{K}_L has fast L^2 off-diagonal decay (Definition 6) then we have $\tau = 2$ in (a).*

Remarks. 1. The upper bound in this theorem follows directly from Corollary 4 (b)

2. In proving this result we will give upper bounds for the value τ when we do not have fast L^2 off-diagonal decay.

3. The kernel corresponding to $\phi(z) = |z|^\alpha/2$ has fast L^2 off-diagonal decay (see the Appendix).

While we have stressed heretofore that our work generalises the known cases in the complex plane, we should point out that these ideas have also been studied on manifolds. For example in [SZ99], [SZ08] and [SZZ05] the authors study the distribution of zeros of random holomorphic sections of (large) powers of a positive holomorphic line bundle L over a compact complex manifold M . Theorem 1, for example, is completely analogous to [SZ99, Theorem 1.1], although our proof is less technical. We have also used many of the ideas from [SZZ05] in our proof of Theorem 4, where the authors also deal with the problem of having no information about a basis. A key difference between the two settings is the compactness of the manifold M , which means that the spaces of sections considered are finite dimensional with a control on the growth of the dimension. There are also some recent results in a non-compact setting, see [DMS12], however the spaces considered are still assumed to be finite dimensional.

The paper is structured as follows: In Section 2 we give some technical results that shall be used throughout the paper, and show that the covariance kernel for the GAFs defined via bases and frames satisfy similar size estimates. In Section 3 we prove Theorems 1 and 2. In Section 4 we show that the smooth linear statistics are asymptotically normal, under some extra regularity assumptions (Theorem 5). In Section 5 we prove the large deviations estimates, Theorem 3 and Corollary 4. In Section 6 we prove Theorem 6, the upper bound for the hole probability for the zero set of the GAF defined via bases. Finally in Section 7 we compute the hole probability for the zero set of the GAF defined via frames, Theorem 7.

The notation $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$. We frequently ignore events of probability zero.

2. DOUBLING MEASURES AND FOCK SPACES

2.1. Technical Preliminaries. We will always assume that μ is a doubling measure (Definition 1) and that ϕ is a subharmonic function with $\mu = \Delta\phi$. Recall that $\rho(z) = \rho_\mu(z)$ is the radius such that $\mu(D(z, \rho_\mu(z))) = 1$. Note that all of the constants (including implicit constants) in this section depend only on the doubling constant C_μ .

Lemma 8. [Chr91, Lemma 2.1] *Let μ be a doubling measure in \mathbb{C} . There exists $\gamma > 0$ such that for any discs D, D' of respective radius $r(D) > r(D')$ with $D \cap D' \neq \emptyset$*

$$\left(\frac{\mu(D)}{\mu(D')} \right)^\gamma \lesssim \frac{r(D)}{r(D')} \lesssim \left(\frac{\mu(D)}{\mu(D')} \right)^{1/\gamma}.$$

In particular, the support of μ has positive Hausdorff dimension. We will sometimes require some further regularity on the measure μ , so we make the following definition.

Definition 3. We say that a doubling measure μ is *locally flat* if given any disc D of radius $r(D)$ satisfying $\mu(D) = 1$ then for every disc $D' \subseteq D$ of radius $r(D')$ we have

$$\frac{1}{\mu(D')} \simeq \left(\frac{r(D)}{r(D')} \right)^2$$

where the implicit constants depend only on μ .

Trivially $\phi(z) = |z|^2$ gives us a locally flat measure, indeed the condition $0 < c < \Delta\phi < C$ ensures that the measure $\Delta\phi$ is locally flat. Moreover there is always a regularisation of the measure $\Delta\phi$ that is locally flat (see [MMO03, Theorem 14]), but we shall not pursue this line of thought.

We have the following estimates from [MMO03, p. 869]: There exist $\eta > 0$, $C > 0$ and $\beta \in (0, 1)$ such that

$$(6) \quad C^{-1}|z|^{-\eta} \leq \rho(z) \leq C|z|^\beta \text{ for } |z| > 1$$

and

$$|\rho(z) - \rho(\zeta)| \leq |z - \zeta| \text{ for } z, \zeta \in \mathbb{C}.$$

Thus ρ is a Lipschitz function, and so in particular is continuous. We will write

$$D^r(z) = D(z, r\rho(z))$$

and

$$D(z) = D^1(z).$$

A simple consequence of Lemma 8 is that $\rho(z) \simeq \rho(\zeta)$ for $\zeta \in D(z)$. We shall make use of the following estimate.

Lemma 9 ([Chr91, p. 205]). *If $\zeta \notin D(z)$ then*

$$\frac{\rho(z)}{\rho(\zeta)} \lesssim \left(\frac{|z - \zeta|}{\rho(\zeta)} \right)^{1-t}$$

for some $t \in (0, 1)$ depending only on the doubling constant, C_μ .

We will need the following estimate.

Lemma 10. [Chr91, Lemma 2.3] *There exists $C > 0$ depending on C_μ such that for any $r > 0$*

$$\int_{D(z,r)} \log \left(\frac{2r}{|z - \zeta|} \right) d\mu(\zeta) \leq C \mu(D(z, r)) \quad z \in \mathbb{C}.$$

We recall that ρ^{-2} can be seen as a regularisation of μ . We define d_μ to be the distance induced by the metric $\rho(z)^{-2} dz \otimes d\bar{z}$, that is

$$d_\mu(z, \zeta) = \inf \int_0^1 |\gamma'(t)| \rho^{-1}(\gamma(t)) dt,$$

where the infimum is taken over all piecewise \mathcal{C}^1 curves $\gamma : [0, 1] \rightarrow \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = \zeta$. We have the following estimates:

Lemma 11 ([MMO03, Lemma 4]). *There exists $\delta > 0$ such that for every $r > 0$ there exists $C_r > 0$ such that*

- $C_r^{-1} \frac{|z - \zeta|}{\rho(z)} \leq d_\mu(z, \zeta) \leq C_r \frac{|z - \zeta|}{\rho(z)}$ if $|z - \zeta| \leq r\rho(z)$ and
- $C_r^{-1} \left(\frac{|z - \zeta|}{\rho(z)} \right)^\delta \leq d_\mu(z, \zeta) \leq C_r \left(\frac{|z - \zeta|}{\rho(z)} \right)^{2-\delta}$ if $|z - \zeta| > r\rho(z)$.

Definition 4. A sequence Λ is ρ -separated if there exists $\delta > 0$ such that

$$\inf_{\lambda \neq \lambda'} d_\mu(\lambda, \lambda') > \delta.$$

One consequence of Lemma 11 is that a sequence Λ is ρ -separated if and only if there exists $\delta > 0$ such that

$$|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda')) \quad \lambda \neq \lambda'.$$

This equivalent condition is often easier to work with and is the reason we call this condition ρ -separated. We shall make repeated use of the following lemma.

Lemma 12. *Let Λ be a ρ -separated sequence. Then for any $\epsilon > 0$ and $k \geq 0$ there exists a constant $C > 0$ depending only on k , ϵ , and C_μ such that*

$$\begin{aligned} \text{(a)} \quad & \int_{\mathbb{C}} \frac{|z - \zeta|^k}{\exp d_\mu^\epsilon(z, \zeta)} \frac{dm(z)}{\rho(z)^2} \leq C \rho^k(\zeta) \text{ and} \\ \text{(b)} \quad & \sum_{\lambda \in \Lambda} \frac{|z - \lambda|^k}{\exp d_\mu^\epsilon(z, \lambda)} \leq C \rho^k(\zeta). \end{aligned}$$

Proof. The proof of (a) is almost identical to [MO09, Lemma 2.7]. Lemma 11 implies that there exists $\epsilon > 0$ such that

$$\exp d_\mu^\epsilon(z, \zeta) \gtrsim \exp \left(\frac{|z - \zeta|}{\rho(\zeta)} \right)^\epsilon.$$

Let $f(x) = x^{\frac{k}{\epsilon}} - \frac{k}{\epsilon} x^{\frac{k}{\epsilon}-1}$ and note that for any $y > 0$

$$\int_y^{+\infty} e^{-x} f(x) dx = e^{-y} y^{k/\epsilon}.$$

Splitting the integral over the regions $D(\zeta)$ and $\mathbb{C} \setminus D(\zeta)$ and using Lemma 9 we see that

$$\begin{aligned} \int_{\mathbb{C}} \frac{|z - \zeta|^k}{\exp d_\mu^\epsilon(z, \zeta)} \frac{dm(z)}{\rho(z)^2} &\lesssim \rho^k(\zeta) + \int_{\mathbb{C} \setminus D(\zeta)} \rho^k(\zeta) \int_{\left(\frac{|z - \zeta|}{\rho(\zeta)}\right)^\epsilon}^{\infty} e^{-x} f(x) dx \frac{dm(z)}{\rho(z)^2} \\ &\lesssim \rho^k(\zeta) + \rho^k(\zeta) \int_1^{+\infty} e^{-x} f(x) \int_{D^{x^{1/\epsilon}}(\zeta)} \frac{dm(z)}{\rho(z)^2} dx \\ &\lesssim \rho^k(\zeta) \left(1 + \int_1^{+\infty} e^{-x} f(x) x^\alpha dx \right) \end{aligned}$$

for some positive α .

We may estimate the sum appearing in (b) by the integral in (a) so the result follows. ■

We will also need the following estimates.

Lemma 13. [MMO03, Lemma 19] *For any $r > 0$ there exists $C = C(r) > 0$ such that for any $f \in \mathcal{H}(\mathbb{C})$ and $z \in \mathbb{C}$:*

$$\begin{aligned} \text{(a)} \quad & |f(z)|^2 e^{-2\phi(z)} \leq C \int_{D^r(z)} |f(\zeta)|^2 e^{-2\phi(\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)}. \\ \text{(b)} \quad & |\nabla(|f|e^{-\phi})(z)|^2 \leq \frac{C}{\rho^2(z)} \int_{D^r(z)} |f(\zeta)|^2 e^{-2\phi(\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)}. \\ \text{(c)} \quad & \text{If } s > r, |f(z)|^2 e^{-2\phi(z)} \leq C_{r,s} \int_{D^s(z) \setminus D^r(z)} |f(\zeta)|^2 e^{-2\phi(\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)}. \end{aligned}$$

We shall scale the measure μ by a (large) parameter $L \geq 1$. We shall write $\phi_L = L\phi$, $\rho_L = \rho_{L\mu}$, $d_L = d_{L\mu}$ and $D_L^r(z) = D(z, r\rho_L(z))$. Note that the measures μ and $L\mu$ have the same doubling

constant, so we may apply all of the results in this section to the measure $L\mu$ without changing the constants. It is clear from the definition

$$L\mu(D(z, \rho_L(z))) = 1$$

that $\rho_L(z) < \rho(z)$ for $L > 1$. Thus by Lemma 8 we have

$$(7) \quad L^\gamma \lesssim \frac{\rho(z)}{\rho_L(z)} \lesssim L^{1/\gamma}$$

and

$$L^{-1/\gamma} \lesssim \frac{d(z, w)}{d_L(z, w)} \lesssim L^{-\gamma}$$

for some $\gamma \leq 1$, where the implicit constants are uniform in z .

If the measure μ is locally flat then we see that

$$(8) \quad \frac{\rho(z)}{\rho_L(z)} \simeq \sqrt{L}$$

and

$$\frac{d(z, w)}{d_L(z, w)} \simeq \frac{1}{\sqrt{L}}.$$

2.2. Kernel estimates. In this section we show that the covariance kernel K_L for the GAF defined via frames (4) satisfies similar growth estimates to the reproducing kernel \mathcal{K}_L , which is the covariance kernel for the GAF defined via bases (3). We will do this by showing that it is the reproducing kernel for a different (but equivalent) norm on the space \mathcal{F}_L^2 . We shall state and prove these results for the function $K = K_1$, but since the constants appearing depend only on the doubling constant, they may be applied *mutatis mutandis* to K_L .

We first note that \mathcal{K} satisfies the following estimates.

Proposition 14 ([MMO03, Lemma 21],[MO09, Theorem 1.1 and Proposition 2.11],[CO11, p. 355]). *There exist positive constants C and ϵ (depending only on the doubling constant for μ) such that for any $z, w \in \mathbb{C}$*

- (a) $|\mathcal{K}(z, w)| \leq C e^{\phi(z)+\phi(w)} e^{-d_\mu^c(z, w)},$
- (b) $C^{-1} e^{2\phi(z)} \leq \mathcal{K}(z, z) \leq C e^{2\phi(z)},$
- (c) $C^{-1}/\rho(z)^2 \leq \Delta \log \mathcal{K}(z, z) \leq C/\rho(z)^2.$
- (d) *There exists $r > 0$ such that $|\mathcal{K}(z, w)| \geq C e^{\phi(z)+\phi(w)}$ for all $w \in D^r(z)$.*

Remark. The off diagonal decay estimates in Theorem 1.1 and Proposition 2.11 of [MO09] differ from the results just stated by factors involving ρ . This is because the authors study spaces with a different norm; in [MMO03, Section 2.3] it is shown that the class of spaces considered here and in [MO09] is the same. However one can easily verify that minor modifications to the proof in [MO09] give the result just stated in the spaces we are considering.

In order to ensure that the sequence of normalised reproducing kernels $(k_\lambda)_{\lambda \in \Lambda}$ form a frame, we require that the sequence Λ is sampling. Sampling sequences in the Fock spaces we consider have been characterised in terms of a Beurling-type density. The following definitions appear in [MMO03].

Definition 5. A sequence Λ is *sampling* for \mathcal{F}_ϕ^2 if there exists $C > 0$ such that for every $f \in \mathcal{F}_\phi^2$

$$(9) \quad C^{-1} \sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\phi(\lambda)} \leq \|f\|_{\mathcal{F}_\phi^2}^2 \leq C \sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\phi(\lambda)}.$$

Theorem 15 ([MMO03, Theorem A]). *A sequence Λ is sampling for \mathcal{F}_ϕ^2 if and only if Λ is a finite union of ρ -separated sequences containing a ρ -separated subsequence Λ' such that $\mathcal{D}_\mu^-(\Lambda') > 1/(2\pi)$ where*

$$\mathcal{D}_\mu^-(\Lambda) = \liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{C}} \frac{\#(\Lambda \cap \overline{D(z, r\rho(z))})}{\mu(D(z, r\rho(z)))} > \frac{1}{2\pi}.$$

Recall that $k_\zeta(z) = \frac{\mathcal{K}(z, \zeta)}{\mathcal{K}(\zeta, \zeta)^{1/2}}$. It is clear from Proposition 14 that $|\langle k_\lambda, f \rangle| \simeq |f(\lambda)|e^{-\phi(\lambda)}$ for all $f \in \mathcal{F}_\phi^2$. Thus Λ is a sampling sequence for \mathcal{F}_ϕ^2 if and only if

$$\|f\|_{\mathcal{F}_\phi^2}^2 \simeq \sum_{\lambda \in \Lambda} |\langle k_\lambda, f \rangle|^2 \quad \text{for all } f \in \mathcal{F}_\phi^2,$$

that is, if and only if $(k_\lambda)_{\lambda \in \Lambda}$ is a frame in \mathcal{F}_ϕ^2 .

We denote the (canonical) dual frame by $(\tilde{k}_\lambda)_{\lambda \in \Lambda}$, and note that any $f \in \mathcal{F}_\phi^2$ can be expanded as

$$f = \sum_{\lambda \in \Lambda} \langle f, \tilde{k}_\lambda \rangle k_\lambda.$$

We introduce a new inner product on the space \mathcal{F}_ϕ^2 given by

$$\langle\langle f, g \rangle\rangle = \sum_{\lambda \in \Lambda} \langle f, \tilde{k}_\lambda \rangle \overline{\langle g, \tilde{k}_\lambda \rangle}$$

and note that the norm $\|f\| = \langle\langle f, f \rangle\rangle^{1/2}$ is equivalent to the original norm $\|\cdot\|_{\mathcal{F}_\phi^2}$ (if Λ is sampling).

Proposition 16. *The reproducing kernel for the (re-normed) space $(\mathcal{F}_\phi^2, \|\cdot\|)$ is*

$$K(z, w) = \sum_{\lambda \in \Lambda} k_\lambda(z) \overline{k_\lambda(w)}.$$

Proof. It is clear that, for each fixed $w \in \mathbb{C}$, $K(\cdot, w) = K_w$ is in the space, so we need only verify the reproducing property. Note first that $\langle k_{\lambda'}, \tilde{k}_\lambda \rangle = \langle \tilde{k}_{\lambda'}, k_\lambda \rangle$. This follows from the fact that $\tilde{k}_\lambda = S^{-1}k_\lambda$, where S is the frame operator associated to $(k_\lambda)_{\lambda \in \Lambda}$, and S is self adjoint with

respect to $\langle \cdot, \cdot \rangle$. Now, for any $f \in \mathcal{F}_\phi^2$,

$$\begin{aligned}
\langle\langle f, K_w \rangle\rangle &= \sum_{\lambda \in \Lambda} \langle f, \tilde{k}_\lambda \rangle \overline{\langle K_w, \tilde{k}_\lambda \rangle} = \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} \langle f, \tilde{k}_\lambda \rangle \overline{\langle k_{\lambda'}, \tilde{k}_\lambda \rangle} k_{\lambda'}(w) \\
&= \sum_{\lambda' \in \Lambda} k_{\lambda'}(w) \sum_{\lambda \in \Lambda} \langle f, \tilde{k}_\lambda \rangle \langle k_\lambda, \tilde{k}_{\lambda'} \rangle \\
&= \sum_{\lambda' \in \Lambda} k_{\lambda'}(w) \left\langle \sum_{\lambda \in \Lambda} \langle f, \tilde{k}_\lambda \rangle k_\lambda, \tilde{k}_{\lambda'} \right\rangle \\
&= \sum_{\lambda' \in \Lambda} k_{\lambda'}(w) \langle f, \tilde{k}_{\lambda'} \rangle = f(w),
\end{aligned}$$

which completes the proof. ■

We now show that the growth and off diagonal diagonal decay of K are similar to that of \mathcal{K} .

Proposition 17. *There exist positive constants C , c and ϵ (depending only on the doubling constant for μ and the sampling constant appearing in (9)) such that for any $z, w \in \mathbb{C}$*

- (a) $|K(z, w)| \leq C e^{\phi(z) + \phi(w)} e^{-c d_\mu^\epsilon(z, w)}$,
- (b) $C^{-1} e^{2\phi(z)} \leq K(z, z) \leq C e^{2\phi(z)}$ and
- (c) $C^{-1} \frac{1}{\rho(z)^2} \leq \Delta \log K(z, z) \leq C \frac{1}{\rho(z)^2}$.
- (d) *There exists $r > 0$ such that $|K(z, w)| \geq C e^{\phi(z) + \phi(w)}$ for all $w \in D^r(z)$.*

Proof. We have (see [Ber70, p. 26])

$$\begin{aligned}
\sqrt{K(z, z)} &= \sup\{|f(z)| : f \in \mathcal{F}_\phi^2, \|f\| \leq 1\} \\
&\simeq \sup\{|f(z)| : f \in \mathcal{F}_\phi^2, \|f\|_{\mathcal{F}_\phi^2} \leq 1\} = \sqrt{\mathcal{K}(z, z)}
\end{aligned}$$

and so Proposition 14 implies (b). Similarly (again see [Ber70, p. 26])

$$\Delta \log K(z, z) = \frac{4 \sup\{|f'(z)|^2 : f \in \mathcal{F}_\phi^2, f(z) = 0; \|f\| \leq 1\}}{K(z, z)} \simeq \Delta \log \mathcal{K}(z, z)$$

so that (c) also follows from Proposition 14.

We note that for all $w \in D^r(z)$, applying Lemma 13(b),

$$||K(w, z)|e^{-\phi(w)} - |K(z, z)|e^{-\phi(z)}| \lesssim \frac{1}{\rho(z)} \|K(\cdot, z)\|_{\mathcal{F}_\phi^2} |z - w| \lesssim r e^{\phi(z)}$$

so that for sufficiently small r , (b) implies (d).

Finally we have, by the estimates in Proposition 14,

$$|K(w, z)| \leq \sum_{\lambda \in \Lambda} |k_\lambda(z) \overline{k_\lambda(w)}| \lesssim e^{\phi(z) + \phi(w)} \sum_{\lambda \in \Lambda} e^{-d_\mu^\epsilon(z, \lambda) - d_\mu^\epsilon(\lambda, w)}.$$

Now

$$\sum_{\lambda \in \Lambda, d_\mu(z, \lambda) > \frac{1}{2}d_\mu(z, w)} e^{-d_\mu^\epsilon(z, \lambda) - d_\mu^\epsilon(\lambda, w)} \leq e^{-2^{-\epsilon}d_\mu^\epsilon(z, w)} \sum_{\lambda \in \Lambda} e^{-d_\mu^\epsilon(\lambda, w)} \lesssim e^{-2^{-\epsilon}d_\mu^\epsilon(z, w)}$$

where we have used Lemma 12. The remaining terms satisfy $d_\mu(w, \lambda) \geq \frac{1}{2}d_\mu(z, w)$ and may be treated similarly. \blacksquare

Remarks. 1. When we apply this result to K_L , it is important that the constants in the relation $\|\cdot\|_L \simeq \|\cdot\|_{\mathcal{F}_L^2}$ are uniform in L , so that the constant C appearing in the conclusion can be taken to be uniform in L . It is not difficult to see that we can always do this. For each L we chose a sequence Λ_L and constants $\delta_0 < R_0$ which do not depend on L satisfying the following properties:

- The discs $(D_L^{\delta_0}(\lambda))_{\lambda \in \Lambda_L}$ are pairwise disjoint.
- We have $\mathbb{C} = \cup_{\lambda \in \Lambda_L} D_L^{R_0}(\lambda)$.
- Each $z \in \mathbb{C}$ is contained in at most N_0 discs of the form $D_L^{R_0+1}(\lambda)$ where N_0 does not depend on z or L .

Applying Lemma 13(b) one can show that if R_0 is sufficiently small then $\sum_{\lambda \in \Lambda_L} |f(\lambda)|^2 e^{-2\phi(\lambda)} \simeq \|f\|_{\mathcal{F}_L^2}^2$ where the implicit constants are uniform in L .

2. We have used only the fact that $(k_\lambda)_{\lambda \in \Lambda}$ is a frame, and the expression of the reproducing kernel as an extremal problem, to show that $K(z, z) \simeq \mathcal{K}(z, z)$ and $\Delta \log K(z, z) \simeq \Delta \log \mathcal{K}(z, z)$. Our proof therefore carries over to any space where these properties hold.

We will sometimes be able to prove sharper results if we assume some extra off-diagonal decay on the kernel \mathcal{K}_L . The condition we will use is the following.

Definition 6. The kernel \mathcal{K}_L has *fast L^2 off-diagonal decay* if, given $C, r > 0$ there exists $R > 0$ (independent of L) such that

$$(10) \quad \sup_{z \in D^r(z_0)} e^{-2\phi_L(z)} \int_{\mathbb{C} \setminus D^R(z_0)} |\mathcal{K}_L(z, \zeta)|^2 e^{-2\phi_L(\zeta)} \frac{dm(\zeta)}{\rho_L(\zeta)^2} \leq e^{-CL}$$

for all $z_0 \in \mathbb{C}$ and L sufficiently large.

Remarks. 1. If $\phi(z) = |z|^2/2$ then since $\mathcal{K}_L(z, \zeta) = e^{Lz\bar{\zeta}}/2\pi^2$ it is easy to see that the \mathcal{K}_L has fast L^2 off-diagonal decay. More generally if $\phi(z) = |z|^\alpha/2$ it can also be seen that \mathcal{K}_L has fast L^2 off-diagonal decay but we postpone the proof to an appendix since it is long and tedious.

2. We also note that [Chr91, Proposition 1.18] shows that there exist ϕ with $0 < c < \Delta\phi < C$ that do not satisfy (10), so that local flatness does not imply fast L^2 decay.

3. PROOF OF THEOREMS 1 AND 2

In this section we will prove Theorems 1 and 2. We will follow the scheme of the proof of [SZ99, Theorem 1.1]. We begin by proving Theorem 1 (a). Recall that n_L is the counting measure on the zero set of the GAF defined via bases, (3).

Proof of Theorem 1(a). Let ψ be a smooth function with compact support in \mathbb{C} . The Edelman-Kostlan formula gives

$$\mathbb{E}[n(\psi, L)] = \frac{1}{4\pi L} \int_{\mathbb{C}} \psi(z) \Delta \log \mathcal{K}_L(z, z) dm(z)$$

so that, by Proposition 14

$$\begin{aligned} \left| \mathbb{E}[n(\psi, L)] - \frac{1}{2\pi} \int \psi d\mu \right| &= \frac{1}{4\pi L} \left| \int_{\mathbb{C}} \Delta \psi(z) (\log \mathcal{K}_L(z, z) - 2\phi_L(z)) dm(z) \right| \\ &\lesssim \frac{1}{L} \int_{\mathbb{C}} |\Delta \psi(z)| dm(z). \end{aligned}$$

■

Proof of Theorem 2. We have (see [SZ08, Theorem 3.3] or [NS11, Lemma 3.3])

$$\mathbb{V}[n(\psi, L)] = \int_{\mathbb{C}} \int_{\mathbb{C}} \Delta \psi(z) \Delta \psi(w) J_L(z, w) dm(z) dm(w)$$

where

$$J_L(z, w) = \frac{1}{16\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{|\mathcal{K}_L(z, w)|^2}{\mathcal{K}_L(z, z) \mathcal{K}_L(z, w)} \right)^n \simeq \frac{|\mathcal{K}_L(z, w)|^2}{\mathcal{K}_L(z, z) \mathcal{K}_L(z, w)}.$$

Fix $z \in \text{supp } \psi$, choose $\alpha > 2/\gamma$ where γ is the constant appearing in (7), and let ϵ be the constant from Proposition 14. Write

$$\begin{aligned} I_1 &= \int_{d_L(z, w) \geq (\alpha \log L)^{1/\epsilon}} \Delta \psi(w) J_L(z, w) dm(w), \\ I_2 &= \int_{d_L(z, w) < (\alpha \log L)^{1/\epsilon}} (\Delta \psi(w) - \Delta \psi(z)) J_L(z, w) dm(w), \\ I_3 &= \int_{d_L(z, w) < (\alpha \log L)^{1/\epsilon}} J_L(z, w) dm(w). \end{aligned}$$

and note that

$$\int_{\mathbb{C}} \Delta \psi(w) J_L(z, w) dm(w) = I_1 + I_2 + \Delta \psi(z) I_3.$$

Now, by Proposition 14, $J_L(z, w) \lesssim e^{-d_L^\epsilon(z, w)} \leq L^{-\alpha}$ when $d_L(z, w) \geq (\alpha \log L)^{1/\epsilon}$ and so

$$|I_1| \leq L^{-\alpha} \int_{d_L(z, w) \geq (\alpha \log L)^{1/\epsilon}} |\Delta \psi(w)| dm(w) \leq L^{-\alpha} \|\Delta \psi\|_{L^1(\mathbb{C})}.$$

Also, since $d_L(z, w) \gtrsim L^\gamma d_\mu(z, w)$, we see that if z and w satisfy $d_L(z, w) < (\alpha \log L)^{1/\epsilon}$ then

$$\Delta \psi(w) - \Delta \psi(z) \rightarrow 0 \text{ as } L \rightarrow \infty,$$

and so

$$|I_2| \leq \sup_{d_L(z, w) < (\alpha \log L)^{1/\epsilon}} |\Delta \psi(w) - \Delta \psi(z)| I_3 = o(I_3).$$

Finally, using Proposition 14 and Lemma 11, we see that

$$I_3 \lesssim \int_{\mathbb{C}} e^{-d_L^e(z,w)} dm(w) \lesssim \int_{\mathbb{C}} e^{-c(\frac{|z-w|}{\rho_L(z)})^{\epsilon'}} dm(w) = \rho_L(z)^2 \int_{\mathbb{C}} e^{-c'|\zeta|^{\epsilon'}} dm(\zeta).$$

Similarly, for r sufficiently small,

$$I_3 \gtrsim \int_{D_L^r(z)} dm(w) = \pi r^2 \rho_L(z)^2,$$

that is, $I_3 \simeq \rho_L(z)^2$. We thus conclude that (note that $\rho_L(z)^2 \gtrsim L^{-2/\gamma} \rho(z)^2 \gtrsim L^{-\alpha}$)

$$\mathbb{V}[n(\psi, L)] = \int_{\mathbb{C}} \Delta\psi(z) (I_1 + I_2 + \Delta\psi(z)I_3) dm(z) \simeq \int_{\mathbb{C}} \Delta\psi(z)^2 \rho_L(z)^2 dm(z)$$

which completes the proof. ■

We will now use the results we have just proved for the mean and the variance of the ‘smooth linear statistics’ to prove Theorem 1 (b).

Proof of Theorem 1 (b). First note that

$$\begin{aligned} \mathbb{E} \left[\left(n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \right)^2 \right] &\lesssim \mathbb{E} \left[\left(n(\psi, L) - \mathbb{E}[n(\psi, L)] \right)^2 \right] \\ &\quad + \left(\mathbb{E}[n(\psi, L)] - \frac{1}{2\pi} \int \psi d\mu \right)^2. \end{aligned}$$

Now Theorem 2 implies that

$$\begin{aligned} \mathbb{E} \left[\left(n(\psi, L) - \mathbb{E}[n(\psi, L)] \right)^2 \right] &= \mathbb{V}[n(\psi, L)] \\ &\simeq L^{-2} \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho_L(z)^2 dm(z) \\ &\lesssim L^{-2(1+\gamma)} \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho(z)^2 dm(z) \end{aligned}$$

while (a) implies that

$$\mathbb{E}[n(\psi, L)] - \frac{1}{2\pi} \int \psi d\mu = O(L^{-1}).$$

We thus infer that

$$\mathbb{E} \left[\left(n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \right)^2 \right] \lesssim L^{-2}$$

which implies that

$$\mathbb{E} \left[\sum_{L=1}^{\infty} \left(\frac{1}{L} n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \right)^2 \right] = \sum_{L=1}^{\infty} \mathbb{E} \left[\left(\frac{1}{L} n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \right)^2 \right] < +\infty.$$

This means that

$$\frac{1}{L} n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \rightarrow 0$$

almost surely, as claimed. ■

4. ASYMPTOTIC NORMALITY

This section consists of the proof of Theorem 5. As we have previously noted, we shall consider only the GAF defined via frames (4). All of the results stated here apply equally well to the GAF defined via bases (3), and the proofs are identical except that the estimates from Proposition 17 should be replaced by the estimates from Proposition 14. Our proof of Theorem 5 is based entirely on the following result which was used to prove asymptotic normality in the case $\phi(z) = |z|^2$ ([ST04, Main Theorem]).

Theorem 18 ([ST04, Theorem 2.2]). *Suppose that for each natural number m , f_m is a Gaussian analytic function with covariance kernel Ξ_m satisfying $\Xi_m(z, z) = 1$ and let n_m be counting measure on the set of zeroes of f_m . Let ν be a measure on \mathbb{C} satisfying $\nu(\mathbb{C}) = 1$ and suppose $\Theta : \mathbb{C} \rightarrow \mathbb{R}$ is a bounded measurable function. Define $Z_m = \int_{\mathbb{C}} \log(|f_m(z)|) \Theta(z) d\nu(z)$ and suppose that*

$$(11) \quad \liminf_{m \rightarrow \infty} \frac{\int \int_{\mathbb{C}^2} |\Xi_m(z, w)|^2 \Theta(z) \Theta(w) d\nu(z) d\nu(w)}{\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} |\Xi_m(z, w)|^2 d\nu(w)} > 0,$$

and that

$$(12) \quad \lim_{m \rightarrow \infty} \sup_{z \in \mathbb{C}} \int_{\mathbb{C}} |\Xi_m(z, w)| d\nu(w) = 0.$$

Then the distributions of the random variables

$$\frac{Z_m - \mathbb{E}Z_m}{\sqrt{\mathbb{V}Z_m}}$$

converge weakly to $\mathcal{N}(0, 1)$ as $m \rightarrow \infty$.

Remark. In fact in [ST04] the authors prove a more general result, but we shall only require the form we have stated. We have also slightly modified the denominator in condition (11), but it is easy to verify that this does not affect the proof (cf. [ST04, Section 2.5]).

Proof of Theorem 5. We consider instead the random variable $n(\psi, L) = \int \psi dn_L$ since it is clear that the factor L^{-1} is unimportant. We first note that Green's formula implies that

$$n(\psi, L) = \frac{1}{2\pi} \int_{\mathbb{C}} \Delta \psi(z) \log |f_L(z)| dm(z)$$

which combined with the Edelman-Kostlan formula gives

$$Z_L(\psi) = n(\psi, L) - \mathbb{E}[n(\psi, L)] = \frac{1}{2\pi} \int_{\mathbb{C}} \Delta \psi(z) \log \frac{|f_L(z)|}{K_L(z, z)^{1/2}} dm(z).$$

Write $\hat{f}_L(z) = \frac{f_L(z)}{K_L(z,z)^{1/2}}$, $\Theta(z) = \frac{c}{2\pi} \Delta\psi(z) \rho(z)^2$ and $d\nu(z) = \frac{1}{c} \chi_{\text{supp } \psi}(z) \frac{dm(z)}{\rho(z)^2}$ where the constant c is chosen so that $\nu(\mathbb{C}) = 1$. Note that

$$Z_L(\psi) = \int_{\mathbb{C}} \log |\hat{f}_L(z)| \Theta(z) d\nu(z)$$

so we need only check that conditions (11) and (12) hold to show asymptotic normality. Here $\Xi_L(z, w) = \frac{K_L(z, w)}{K_L(z, z)^{1/2} K_L(w, w)^{1/2}}$. Now, by the estimates of Proposition 17 and (8),

$$\begin{aligned} \int_{\mathbb{C}} |\Xi_L(z, w)| d\nu(w) &\simeq e^{-\phi_L(z)} \int_{\mathbb{C}} |K_L(z, w)| e^{-\phi_L(w)} d\nu(w) \\ &\leq e^{-\phi_L(z)} \left(\int_{\mathbb{C}} |K_L(z, w)|^2 e^{-2\phi_L(w)} \frac{dm(w)}{\rho(w)^2} \right)^{1/2} \nu(\mathbb{C})^{1/2} \\ &\stackrel{(*)}{\simeq} L^{-\frac{1}{2}} e^{-\phi_L(z)} \left(\int_{\mathbb{C}} |K_L(z, w)|^2 e^{-2\phi_L(w)} \frac{dm(w)}{\rho_L(w)^2} \right)^{1/2} \simeq L^{-\frac{1}{2}}, \end{aligned}$$

where we have used local flatness (8) for the estimate (*), so (12) holds. (In fact to prove (12) it suffices to use the estimate (7).) Similarly

$$\int_{\mathbb{C}} |\Xi_L(z, w)|^2 d\nu(w) \simeq L^{-1}.$$

By a computation almost identical to that in the proof of Theorem 2 we also have

$$\begin{aligned} \iint_{\mathbb{C}^2} |\Xi_L(z, w)|^2 \Theta(z) \Theta(w) d\nu(z) d\nu(w) &\simeq \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho_L(z)^2 dm(z) \\ &\simeq L^{-1} \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho(z)^2 dm(z) \end{aligned}$$

which verifies (11). (In both of these estimates we use (8) since the estimate (7) is not enough, it is here that our local flatness assumption is important.) \blacksquare

5. LARGE DEVIATIONS

In this section we prove Theorem 3 and Corollary 4. We borrow many of the ideas used here from [ST05] and [SZZ05], but some modifications are necessary to deal with the fact that ϕ is non-radial and we are in a non-compact setting. The key ingredient in the proof of Theorem 3 is the following lemma.

Lemma 19. *For any disc $D = D^r(z_0)$ and any $\delta > 0$ there exists $c > 0$ depending only on δ , D and μ such that*

$$\int_D |\log |f_L(z)| - \phi_L(z)| dm(z) \leq \delta L$$

outside an exceptional set of probability at most e^{-cL^2} , for L sufficiently large.

We begin with the following lemma.

Lemma 20. *Given a disc $D = D^r(z_0)$ and $\delta > 0$ there exists $c > 0$ depending only on the doubling constant such that*

$$\left| \max_{z \in \overline{D}} (\log |f_L(z)| - \phi_L(z)) \right| \leq \delta L$$

outside an exceptional set of probability at most $e^{-c\delta\mu(D)L^2}$, for L sufficiently large.

Proof. Define $\hat{f}_L(z) = \frac{f_L(z)}{K_L(z, z)^{1/2}}$. We will show that

$$\mathbb{P} \left[\left| \max_{z \in \overline{D}} \log |\hat{f}_L(z)| \right| \geq \delta L \right] \leq e^{-c\delta\mu(D)L^2}$$

for L sufficiently large, which will imply the claimed result by Proposition 17(b). We divide the proof in two parts.

1. We first show that

$$\mathbb{P} \left[\max_{z \in \overline{D}} |\hat{f}_L(z)| \leq e^{-\delta L} \right] \leq e^{-c\delta\mu(D)L^2}.$$

For each L define S_L to be a ρ_L -separated sequence with the constant

$$R = \inf \{d_L(s, t) : s \neq t \text{ and } s, t \in S_L\}$$

to be chosen (large but uniform in L). Moreover we assume that

$$\sup_{z \in \mathbb{C}} d_L(z, S_L) < \infty,$$

uniformly in L once more. Trivially

$$\mathbb{P} \left[\max_{z \in \overline{D}} |\hat{f}_L(z)| \leq e^{-\delta L} \right] \leq \mathbb{P} \left[|\hat{f}_L(s)| \leq e^{-\delta L} \text{ for all } s \in \overline{D} \cap S_L \right]$$

and we now estimate the probability of this event. We write

$$\overline{D} \cap S_L = \{s_1, \dots, s_N\}.$$

Note that for R_1 sufficiently small

$$L\mu(D^{2r}(z_0)) \leq \sum_{j=1}^N L\mu(D_L^{R_1}(s_j)) \lesssim N$$

while for R_2 large enough

$$L\mu(D^r(z_0)) \geq \sum_{j=1}^N L\mu(D_L^{R_2}(s_j)) \gtrsim N$$

so that $N \simeq L\mu(D)$. Consider the vector

$$\xi = \begin{pmatrix} \hat{f}_L(s_1) \\ \vdots \\ \hat{f}_L(s_N) \end{pmatrix}$$

which is a mean-zero N -dimensional complex normal with covariance matrix σ given by

$$\sigma_{mn} = \frac{K_L(s_m, s_n)}{K_L(s_m, s_m)^{1/2} K_L(s_n, s_n)^{1/2}}.$$

Note that $\sigma_{nn} = 1$ and $|\sigma_{mn}| \lesssim e^{-d_L^\epsilon(s_n, s_m)}$ so that if the sequence S_L is chosen to be sufficiently separated then the components of the vector ξ will be ‘almost independent’. We write $\sigma = I + A$ and note that

$$\begin{aligned} \max_n \left| \sum_{m \neq n} \sigma_{mn} \right| &\lesssim \max_n \sum_{m \neq n} e^{-d_L^\epsilon(s_n, s_m)} \lesssim \max_n \int_{\mathbb{C} \setminus B_L(s_n, R)} e^{-d_L^\epsilon(s_n, w)} \frac{dm(w)}{\rho_L(w)^2} \\ &\lesssim \int_{R^{\epsilon'}}^\infty x^\alpha e^{-x} dx \end{aligned}$$

for some $\alpha, \epsilon' > 0$ by an argument identical to that given in the proof of Lemma 12. Thus by choosing R sufficiently large we have $\|A\|_\infty \leq \frac{1}{2}$ and so for any $v \in \mathbb{C}^N$

$$\|\sigma v\|_\infty \geq \frac{1}{2} \|v\|_\infty.$$

Thus the eigenvalues of σ are bounded below by $\frac{1}{2}$ and so if $\sigma = BB^*$ then

$$\|B^{-1}\|_2 \leq \sqrt{2}.$$

Now the components of the vector $\zeta = B^{-1}\xi$ are iid $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables, which we denote ζ_j , and moreover

$$\|\zeta\|_\infty \leq \|B^{-1}\xi\|_2 \leq \sqrt{2} \|\xi\|_2 \leq \sqrt{2N} \|\xi\|_\infty.$$

This means that

$$\begin{aligned} \mathbb{P} \left[|\hat{f}_L(s)| \leq e^{-\delta L} \text{ for all } s \in \overline{D} \cap S_L \right] &\leq \mathbb{P} \left[|\zeta_j| \leq \sqrt{2N} e^{-\delta L} \text{ for all } 1 \leq j \leq N \right] \\ &= (1 - \exp(-2N e^{-2\delta L}))^N \leq e^{-c\delta \mu(D)L^2} \end{aligned}$$

for L sufficiently large, where c depends only on the doubling constant (and $\sup_{z \in \mathbb{C}} d_L(z, S_L)$), as claimed.

2. For the second part of the proof we must estimate

$$\mathbb{P}[\max_{z \in \overline{D}} |\hat{f}_L(z)| \geq e^{\delta L}]$$

and so we define the event

$$\mathcal{E} = \{\max_{z \in \overline{D}} |\hat{f}_L(z)| \geq e^{\delta L}\}.$$

We write $\tilde{\Lambda}_L = \Lambda_L \cap D^{2r}(z_0)$ and $\tilde{f}_L = \sum_{\lambda \in \tilde{\Lambda}_L} a_\lambda k_\lambda(z)$, and note that $\#\tilde{\Lambda}_L \simeq L\mu(D)$ as in the first part of the proof. Consider the event

$$\mathcal{A} = \{|a_\lambda| \leq L \frac{|\lambda - z_0|}{\rho_L(z_0)} \text{ for } \lambda \in \Lambda_L \setminus \tilde{\Lambda}_L\}.$$

If the event \mathcal{A} occurs and $z \in \overline{D}$ then, since $d_L(\lambda, z) \geq C_r d_L(\lambda, z_0)$ for some $C_r > 0$, we have by Proposition 17(a)

$$\begin{aligned} \left| \frac{f_L(z) - \tilde{f}_L(z)}{K_L(z, z)^{1/2}} \right| &\lesssim \sum_{\lambda \in \Lambda_L \setminus \tilde{\Lambda}_L} |a_\lambda| e^{-d_L^\epsilon(\lambda, z)} \\ &\leq \frac{L}{\rho_L(z_0)} \sum_{\lambda \in \Lambda_L \setminus \tilde{\Lambda}_L} |\lambda - z_0| e^{-C_r^\epsilon d_L^\epsilon(\lambda, z)} \\ &\lesssim \frac{L}{\rho_L(z_0)} \int_{\mathbb{C} \setminus D^r(z_0)} |\zeta - z_0| e^{-C_r^\epsilon d_L^\epsilon(\zeta, z)} \frac{dm(\zeta)}{\rho_L(\zeta)^2} \lesssim L \end{aligned}$$

where the final estimate comes from an argument similar to that used in the proof of Lemma 12 and the implicit constant depends on r . Hence the event $\mathcal{A} \cap \mathcal{E}$ implies that

$$\max_{z \in \overline{D}} \left| \frac{\tilde{f}_L(z)}{K_L(z, z)^{1/2}} \right| \geq e^{\delta L} - C'_r L \geq e^{\frac{\delta L}{2}}$$

for L sufficiently large, where C'_r is another positive constant. Now a simple application of the Cauchy-Schwartz inequality shows that

$$|\tilde{f}_L(z)| \leq \left(\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda|^2 \right)^{1/2} \left(\sum_{\lambda \in \tilde{\Lambda}_L} |k_\lambda(z)|^2 \right)^{1/2} \leq \left(\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda|^2 \right)^{1/2} K_L(z, z)^{1/2}$$

and so

$$\begin{aligned} \mathbb{P}[\mathcal{A} \cap \mathcal{E}] &\leq \mathbb{P} \left[\max_{z \in \overline{D}} \left| \frac{\tilde{f}_L(z)}{K_L(z, z)^{1/2}} \right| \geq e^{\frac{\delta L}{2}} \right] \\ &\leq \mathbb{P} \left[\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda|^2 \geq e^{\delta L} \right] \\ &\leq \mathbb{P} \left[|a_\lambda|^2 \geq \frac{e^{\delta L}}{\#\tilde{\Lambda}_L} \text{ for all } \lambda \in \tilde{\Lambda}_L \right] = \left(\exp - \frac{e^{\delta L}}{\#\tilde{\Lambda}_L} \right)^{\#\tilde{\Lambda}_L} = e^{-e^{\delta L}}. \end{aligned}$$

We finally estimate the probability of the event \mathcal{A} ; using (7) and (6) we see that

$$\begin{aligned}
\log \mathbb{P}[\mathcal{A}] &= \log \prod_{\lambda \in \Lambda_L \setminus \tilde{\Lambda}_L} \left(1 - \exp \left(-L^2 \frac{|\lambda - z_0|^2}{\rho_L^2(z_0)} \right) \right) \\
&\simeq - \sum_{\lambda \in \Lambda_L \setminus \tilde{\Lambda}_L} \exp \left(-L^2 \frac{|\lambda - z_0|^2}{\rho_L^2(z_0)} \right) \\
&\gtrsim - \int_{\mathbb{C} \setminus D} \exp \left(-L^2 \frac{|\zeta - z_0|^2}{\rho_L^2(z_0)} \right) \frac{dm(\zeta)}{\rho_L(\zeta)^2} \\
&\gtrsim -L^{2/\gamma} \int_{\mathbb{C} \setminus D} \exp \left(-CL^{2+2/\gamma} \frac{|\zeta - z_0|^2}{\rho^2(z_0)} \right) \frac{dm(\zeta)}{\rho(\zeta)^2} \\
(13) \quad &\gtrsim -C_0 L^{2/\gamma} e^{-C_1 L^{2+2/\gamma}}
\end{aligned}$$

where C_0 and C_1 depend on r and the doubling constant, and the final estimate uses an argument similar to that given in the proof of Lemma 12. We finally compute that

$$\mathbb{P}[\mathcal{E}] \leq \mathbb{P}[\mathcal{E} \cap \mathcal{A}] + \mathbb{P}[\mathcal{A}^c] \leq e^{-e^{\delta L}} + (1 - \exp\{-C_0 L^{2/\gamma} e^{-C_1 L^{2+2/\gamma}}\}) \leq e^{-cL^2}$$

for L sufficiently large and for any positive c . ■

Lemma 21. *Given a disc $D = D^r(z_0)$ there exist $c, C > 0$ depending only on the doubling constant such that*

$$\int_D |\log |f_L(z)| - \phi_L(z)| dm(z) \leq Cr^2 \rho(z_0)^2 \mu(D) L$$

outside of an exceptional set of probability at most $e^{-c\mu(D)L^2}$, for L sufficiently large.

We will use the following result to prove this lemma.

Theorem 22 ([Pas88, Chap. 1 Lemma 7] or [AC96, Theorem 1]). *If u is a subharmonic function on \mathbb{D} then, for all $\zeta \in \mathbb{D}$,*

$$u(\zeta) = \int_{\mathbb{D}} \tilde{P}(\zeta, z) u(z) dm(z) - \int_{\mathbb{D}} \tilde{G}(\zeta, z) \Delta u(z)$$

where

$$\tilde{P}(\zeta, z) = \frac{1}{\pi} \frac{(1 - |\zeta|^2)^2}{|1 - \bar{z}\zeta|^4}$$

and

$$\tilde{G}(\zeta, z) = \frac{1}{4\pi} \left(\log \left| \frac{1 - \bar{\zeta}z}{\zeta - z} \right|^2 - \left(1 - \left| \frac{\zeta - z}{1 - \bar{\zeta}z} \right|^2 \right) \right)$$

Proof of Lemma 21. Applying Lemma 20 we see that outside of an exceptional set, we may find $\zeta \in D^{r/2}(z_0)$ such that

$$-L\mu(D) \leq \log |f_L(\zeta)| - \phi_L(\zeta).$$

Making the appropriate change of variables in Theorem 22 and applying the resulting decomposition to the subharmonic functions $\log |f_L|$ and ϕ_L on D we see that

$$\begin{aligned} \log |f_L(\zeta)| - \phi_L(\zeta) &= \int_D \tilde{P} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) (\log |f_L(z)| - \phi_L(z)) \frac{dm(z)}{r^2\rho(z_0)^2} \\ &\quad - \int_D \tilde{G} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) (2\pi dn_L(z) - \Delta\phi_L(z)) \\ &\leq \int_D \tilde{P} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) (\log |f_L(z)| - \phi_L(z)) \frac{dm(z)}{r^2\rho(z_0)^2} \\ &\quad + \int_D \tilde{G} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) \Delta\phi_L(z) \end{aligned}$$

since \tilde{G} is always positive. Now, since $\zeta \in D^{r/2}(z_0)$, we have by Lemma 10

$$\begin{aligned} \int_D \tilde{G} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) \Delta\phi_L(z) &\leq \int_{D^{r/2}(\zeta)} \tilde{G} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) \Delta\phi_L(z) \\ &\quad + \int_{D \setminus D^{r/2}(\zeta)} \tilde{G} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)} \right) \Delta\phi_L(z) \\ &\lesssim L \int_{D^{r/2}(\zeta)} \log \left(\frac{\frac{3}{2}r}{|\zeta - z|} \right) d\mu(z) + L\mu(D) \\ &\lesssim L\mu(D) \end{aligned}$$

and so

$$0 \leq \int_D \tilde{P} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{w - z_0}{r\rho(z_0)} \right) (\log |f_L(w)| - \phi_L(w)) \frac{dm(w)}{r^2\rho(z_0)^2} + CL\mu(D)$$

for some positive C depending only on the doubling constant. Noting that \tilde{P} is also positive and satisfies

$$\tilde{P} \left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{w - z_0}{r\rho(z_0)} \right) \simeq 1$$

for $w \in D$ and $\zeta \in D^{r/2}(z_0)$ we see that

$$\int_D \log^- (|f_L(w)| e^{-\phi_L(w)}) \frac{dm(w)}{r^2\rho(z_0)^2} \lesssim \int_D \log^+ (|f_L(w)| e^{-\phi_L(w)}) \frac{dm(w)}{r^2\rho(z_0)^2} + L\mu(D)$$

and so

$$\int_D |\log |f_L(w)| - \phi_L(w)| \frac{dm(w)}{r^2\rho(z_0)^2} \lesssim \int_D \log^+ (|f_L(w)| e^{-\phi_L(w)}) \frac{dm(w)}{r^2\rho(z_0)^2} + L\mu(D).$$

Applying Lemma 20 once more we see that outside of another exceptional set

$$\int_D \log^+ (|f_L(w)| e^{-\phi_L(w)}) \frac{dm(w)}{r^2\rho(z_0)^2} \lesssim L\mu(D)$$

which completes the proof. ■

We are now ready to prove Lemma 19

Proof of Lemma 19. Given $\delta > 0$ we may cover D with discs $(D^{r_j}(z_j))_{j=1}^N$ such that $\mu(D^{r_j}(z_j)) = \delta$ and $z_j \in U$. The Vitali covering lemma implies that we may assume that $N \lesssim \mu(D)/\delta$. Now, applying Lemma 21 we see that outside of an exceptional set

$$\int_U |\log |f_L(z)| - \phi_L(z)| dm(z) \leq \delta L \sum_{j=1}^N r_j^2 \rho(z_j)^2.$$

We finally note that $\rho(z_j) \simeq \rho(z_0)$ and that Lemma 9 implies that

$$r_j \lesssim \delta^\gamma$$

for all j . Thus

$$\int_U |\log |f_L(z)| - \phi_L(z)| dm(z) \lesssim \delta L N \delta^\gamma \lesssim L \delta^{2\gamma}.$$

Appropriately changing the value of δ completes the proof. ■

Proof of Theorem 3. We have already noted that the proof of (a) is identical to the proof of Theorem 1 (a). It remains to show the large deviations estimate (b). We first note that

$$\begin{aligned} \left| n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \right| &= \frac{1}{2\pi L} \left| \int_{\mathbb{C}} \Delta\psi(z) (\log |f_L(z)| - \phi_L(z)) dm(z) \right| \\ &\leq \frac{1}{2\pi L} \max_{z \in \mathbb{C}} |\Delta\psi(z)| \int_{\text{supp } \psi} |\log |f_L| - \phi_L| dm \end{aligned}$$

and so applying Lemma 19 with $\delta' = \delta |\int \psi d\mu| / \|\Delta\psi\|_\infty$ we see that

$$\left| n(\psi, L) - \frac{1}{2\pi} \int \psi d\mu \right| \leq \delta \left| \frac{1}{2\pi} \int \psi d\mu \right|$$

outside an exceptional set of probability at most e^{-cL^2} , as claimed. ■

Proof of Corollary 4. Let $\delta > 0$ and choose smooth, compactly supported ψ_1 and ψ_2 satisfying

$$0 \leq \psi_1 \leq \chi_U \leq \psi_2 \leq 1,$$

$$\int_{\mathbb{C}} \psi_1 d\mu \geq \mu(U)(1 - \delta)$$

and

$$\int_{\mathbb{C}} \psi_2 d\mu \leq \mu(U)(1 + \delta).$$

(a) Applying Theorem 3 (a) we see that, for L sufficiently large,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{L} n_L(U) \right] - \frac{1}{2\pi} \mu(U) &\leq \mathbb{E} \left[\frac{1}{L} \int \psi_2 dn_L \right] - \frac{1}{2\pi} \mu(U) \\ &\leq \frac{1}{2\pi} \int \psi d\mu + \frac{C}{L} \int_{\mathbb{C}} |\Delta \psi_2(z)| dm(z) - \frac{1}{2\pi} \mu(U) \\ &\leq \frac{\delta}{2\pi} \mu(U) + \frac{C}{L} \int_{\mathbb{C}} |\Delta \psi_2(z)| dm(z). \end{aligned}$$

Similarly

$$\mathbb{E} \left[\frac{1}{L} n_L(U) \right] - \frac{1}{2\pi} \mu(U) \geq -\frac{\delta}{2\pi} \mu(U) - \frac{C}{L} \int_{\mathbb{C}} |\Delta \psi_2(z)| dm(z).$$

Choosing first δ small and then L large (depending on δ) completes the proof of (a).

(b) Outside an exceptional set of probability e^{-cL^2} we have, by Theorem 3 (b)

$$\frac{1}{L} n(\psi_2, L) \leq (1 + \delta) \frac{1}{2\pi} \int_{\mathbb{C}} \psi_2 d\mu.$$

We see that

$$\frac{1}{L} n_L(U) \leq \frac{1}{L} n(\psi_2, L) \leq (1 + \delta) \frac{1}{2\pi} \int_{\mathbb{C}} \psi_2 d\mu \leq (1 + \delta)^2 \frac{\mu(U)}{2\pi}$$

whence

$$\frac{\frac{1}{L} n_L(U)}{\frac{\mu(U)}{2\pi}} - 1 \lesssim \delta.$$

Similarly

$$\frac{\frac{1}{L} n_L(U)}{\frac{\mu(U)}{2\pi}} - 1 \gtrsim -\delta.$$

outside another exceptional set of probability e^{-cL^2} , which after appropriately changing the value of δ completes the proof. ■

6. PROOF OF THEOREM 6

We will use some of the ideas from the proof of Theorem 4 here. We begin with a lemma that is very similar to Lemma 20. It is clear that if we could prove an exact analogue of Lemma 20 then we could prove a large deviations theorem, since it is only in the proof of this lemma that we use the decay estimates for the frame elements. Unfortunately we are unable to prove such a result, but the following result will be enough to prove a hole theorem. Recall that we write $D = D^r(z_0)$.

Lemma 23. *Given $z_0 \in \mathbb{C}$ and $\delta, r > 0$ there exists $c > 0$ depending only on the doubling constant such that*

$$\mathbb{P} \left[\max_{z \in D} (\log |g_L(z)| - \phi_L(z)) \leq -\delta L \right] \leq e^{-c\delta\mu(D)L^2}$$

for L sufficiently large. Moreover there exists $C' > 0$ depending on ϕ and r such that for all $z_0 \in \mathbb{C}$ and $C > C'$

$$\mathbb{P} \left[\max_{z \in \overline{D}} (\log |g_L(z)| - \phi_L(z)) \geq CL \right] \leq e^{-cL^2}$$

for sufficiently large L .

Proof. The proof that

$$\mathbb{P} \left[\max_{z \in \overline{D}} (\log |g_L(z)| - \phi_L(z)) \leq -\delta L \right] \leq e^{-c\delta\mu(D)L^2}$$

is identical to the proof of the first part of Lemma 20, we omit the details.

To prove the second estimate we use the following result, which is simply [HKPV09, Lemma 2.4.4] translated and re-scaled.

Lemma 24. *Let f be a Gaussian analytic function in a neighbourhood of the disc $D(z_0, R)$ with covariance kernel K . Then for $r < R/2$ we have*

$$\mathbb{P} \left[\max_{z \in \overline{D(z_0, r)}} |f(z)| > t \right] \leq 2e^{-t^2/8\sigma_{2r}^2}$$

where $\sigma_{2r}^2 = \max\{K(z, z) : z \in \overline{D(z_0, 2r)}\}$.

Let $C_1 = \min\{\phi(z) : z \in \overline{D}\}$ and $C_2 = \max\{\phi(z) : z \in \overline{D^{2r}(z_0)}\}$. Note that

$$\max\{\mathcal{K}_L(z, z) : z \in D(z_0, 2r)\} \lesssim e^{2C_2L}.$$

Hence

$$\begin{aligned} \mathbb{P} \left[\max_{z \in \overline{D}} (\log |g_L(z)| - \phi_L(z)) \geq CL \right] &\leq \mathbb{P} \left[\max_{z \in \overline{D}} |g_L(z)| \gtrsim e^{(C+C_1)L} \right] \\ &\leq 2 \exp\{-c'e^{2(C+C_1-C_2)L}\} \leq e^{-cL^2} \end{aligned}$$

for any $c > 0$ if $C + C_1 - C_2 > 0$. ■

We may now immediately infer the following lemma. All integrals over circles are understood to be with respect to normalised Lebesgue measure on the circle.

Lemma 25. *For any $z_0 \in \mathbb{C}$ and any $\delta, r > 0$ there exists $c > 0$ depending only on $\delta, \Delta\phi(D)$ and the doubling constant such that*

$$\mathbb{P} \left[\int_{\partial D} (\log |g_L(z)| - \phi_L(z)) \leq -\delta L \right] \leq e^{-cL^2}$$

for L sufficiently large.

Proof. It suffices to show this for small δ . Put $\kappa = 1 - \delta^{1/4}$, $N = [2\pi\delta^{-1}]$ and define $z_j = z_0 + \kappa r \rho(z_0) \exp(\frac{2\pi i j}{N})$ and $D_j = D(z_j, \delta r \rho(z_0))$ for $j = 1, \dots, N$. Lemma 23 implies that outside an exceptional set of probability at most $N e^{-c\delta\mu(D_j)L^2} \leq e^{-c'L^2}$ (where c' depends on δ , $\mu(D^r(z_0))$ and the doubling constant) there exist $\zeta_j \in \overline{D_j}$ such that

$$\log |g_L(\zeta_j)| - \phi_L(\zeta_j) \geq -\delta L.$$

Let $P(\zeta, z)$ and $G(\zeta, z)$ be, respectively, the Poisson kernel and the Green function for D where we use the convention that the Green function is positive. Applying the Riesz decomposition to the subharmonic functions $\log |g_L|$ and ϕ_L on the disc D implies that

$$\begin{aligned} -\delta L &\leq \frac{1}{N} \sum_{j=1}^N (\log |g_L(\zeta_j)| - \phi_L(\zeta_j)) \\ &= \int_{\partial D} (\log |g_L(z)| - \phi_L(z)) + \int_{\partial D} \left(\frac{1}{N} \sum_{j=1}^N P(\zeta_j, z) - 1 \right) (\log |g_L(z)| - \phi_L(z)) \\ &\quad - \int_D \frac{1}{N} \sum_{j=1}^N G(\zeta_j, z) (dn_L(z) - \Delta \phi_L(z)) \\ &\leq \int_{\partial D} (\log |g_L(z)| - \phi_L(z)) + \int_{\partial D} \left(\frac{1}{N} \sum_{j=1}^N P(\zeta_j, z) - 1 \right) (\log |g_L(z)| - \phi_L(z)) \\ &\quad + \int_D \frac{1}{N} \sum_{j=1}^N G(\zeta_j, z) \Delta \phi_L(z). \end{aligned}$$

Claim 26. *There exists $\tilde{C} > 0$ such that*

$$\int_{\partial D} |\log |g_L(z)| - \phi_L(z)| \leq \tilde{C} \mu(D) L$$

outside of an exceptional set of probability at most e^{-cL^2} .

Claim 27 ([ST05, Claim 2]). *There exists $C_0 > 0$ such that*

$$\max_{z \in D} \left| \frac{1}{N} \sum_{j=1}^N P(\zeta_j, z) - 1 \right| \leq C_0 \delta^{1/2}$$

Claim 28. *There exists $C_1 > 0$ and $0 < \alpha < 1/4$ depending only on the doubling constant and $\mu(D)$ such that*

$$\int_D G(\zeta_j, z) \Delta \phi_L(z) \leq C_1 \delta^\alpha L$$

for δ sufficiently small.

Applying Claims 26 and 27 we see that outside another exceptional set we have

$$\left| \int_{\partial D} \left(\frac{1}{N} \sum_{j=1}^N P(\zeta_j, z) - 1 \right) (\log |g_L(z)| - \phi_L(z)) \right| \lesssim \delta^{1/2} L$$

while Claim 28 implies that

$$\int_D \frac{1}{N} \sum_{j=1}^N G(\zeta_j, z) \Delta \phi_L(z) \leq C_1 \delta^\alpha L.$$

Hence

$$\int_{\partial D} (\log |g_L(z)| - \phi_L(z)) \geq -(\delta + C_0 \delta^{3/2} + C_1 \delta^\alpha) L \gtrsim -\delta^\alpha L$$

outside an exceptional set, and so the lemma follows. \blacksquare

Proof of Claim 26. We use the same notation. Lemma 23 implies that outside an exceptional set of probability at most e^{-cL^2} there exists $\zeta_0 \in \overline{D^{r/2}(z_0)}$ such that

$$\log |g_L(\zeta_j)| - \phi_L(\zeta_j) \geq -\mu(D)L.$$

Another application of the Riesz decomposition to the subharmonic functions $\log |g_L|$ and ϕ_L on the disc D implies that

$$\begin{aligned} -\mu(D)L &\leq \log |g_L(\zeta_0)| - \phi_L(\zeta_0) \\ &= \int_{\partial D} P(\zeta_0, z) (\log |g_L(z)| - \phi_L(z)) - \int_D G(\zeta_0, z) (2\pi dn_L(z) - \Delta \phi_L(z)) \\ &\leq \int_{\partial D} P(\zeta_0, z) (\log |g_L(z)| - \phi_L(z)) + L \int_D G(\zeta_0, z) \Delta \phi(z). \end{aligned}$$

Now since $\zeta_0 \in D^{r/2}(z_0)$, we have by Lemma 10

$$\begin{aligned} \int_D G(\zeta_0, z) \Delta \phi(z) &\leq \int_{D^{r/2}(\zeta_0)} G(\zeta_0, z) \Delta \phi(z) + \int_{D \setminus D^{r/2}(w)} G(\zeta_0, z) \Delta \phi(z) \\ &\lesssim \int_{D^{r/2}(\zeta_0)} \log \left(\frac{\frac{3}{2}r}{|\zeta_0 - z|} \right) d\mu(z) + \mu(D) \\ &\lesssim \mu(D) \end{aligned}$$

and so

$$0 \leq \int_{\partial D} P(\zeta_0, z) (\log |g_L(z)| - \phi_L(z)) + CL\mu(D)$$

for some positive C depending only on the doubling constant. Thus

$$\int_{\partial D} P(\zeta_0, z) \log^- (|g_L(z)| e^{-\phi_L(z)}) \leq \int_{\partial D} P(\zeta_0, z) \log^+ (|g_L(z)| e^{-\phi_L(z)}) + CL\mu(D).$$

We note that for $z \in \partial D$ and $\zeta_0 \in D^{r/2}(z_0)$ we have

$$\frac{1}{3} \leq P(\zeta_0, z) \leq 3$$

and so

$$\int_{\partial D} |\log |g_L(z)| - \phi_L(z)| \lesssim \int_{\partial D} \log^+(|g_L(z)|e^{-\phi_L(z)}) + L\mu(D).$$

Applying Lemma 20 once more we see that outside of another exceptional set

$$\int_{\partial D} \log^+(|g_L(z)|e^{-\phi_L(z)}) \lesssim L\mu(D)$$

which completes the proof. ■

Proof of Claim 28. To simplify the notation we move to the unit disc. We write $\varphi(w) = \phi(z_0 + r\phi(z_0)w)$ for $w \in \mathbb{D}$ and $w_j = (\zeta_j - z_0)/r\phi(z_0)$ and note that $1 - |w_j| \lesssim \delta^{1/4}$. We see that

$$\int_D G(\zeta_j, z) \Delta \phi_L(z) = \frac{L}{2\pi} \int_{\mathbb{D}} \log \left| \frac{1 - \overline{w_j}w}{w - w_j} \right| \Delta \varphi(w)$$

and we write

$$B_j(r) = \{w \in \mathbb{D} : \left| \frac{w - w_j}{1 - \overline{w_j}w} \right| \leq r\} = D\left(\frac{1 - r^2}{1 - r^2|w_j|^2}w_j, \frac{1 - |w_j|^2}{1 - r^2|w_j|^2}r\right)$$

for the hyperbolic discs of centre w_j and radius r . Fix some $\beta < 1/4$ and note that

$$\int_{\mathbb{D} \setminus B_j(1-\delta^\beta)} \log \left| \frac{1 - \overline{w_j}w}{w - w_j} \right| \Delta \varphi(w) \leq -\log(1 - \delta^\beta) \Delta \varphi(\mathbb{D}) \leq 2\delta^\beta \mu(D).$$

Also, using the distribution function, we see that

$$\begin{aligned} \int_{B_j(1-\delta^\beta)} \log \left| \frac{1 - \overline{w_j}w}{w - w_j} \right| \Delta \varphi(w) &= \int_0^\infty \Delta \varphi(B_j(1 - \delta^\beta) \cap B_j(e^{-x})) dx \\ &= \int_0^{-\log(1-\delta^\beta)} \Delta \varphi(B_j(1 - \delta^\beta)) dx \\ &\quad + \int_{-\log(1-\delta^\beta)}^\infty \Delta \varphi(B_j(e^{-x})) dx \\ &\leq 2\delta^\beta \mu(D) + \int_{-\log(1-\delta^\beta)}^\infty \Delta \varphi(B_j(e^{-x})) dx. \end{aligned}$$

Now the Euclidean radius of the disc $B_j(e^{-x})$ is given by

$$\frac{1 - |w_j|^2}{1 - e^{-2x}|w_j|^2} e^{-x} \lesssim \frac{1 - |w_j|}{1 - e^{-x}|w_j|} \lesssim \frac{\delta^{1/4}}{\delta^\beta}$$

which gets arbitrarily small, while $\rho_{\Delta\varphi}(w) \simeq \rho_{\Delta\varphi}(0)$ for all $w \in \mathbb{D}$. Applying Lemma 9 to the doubling measure $\Delta\varphi$ we see that there exists $0 < \gamma < 1$ such that

$$\begin{aligned} \int_{-\log(1-\delta^\beta)}^{\infty} \Delta\varphi(B_j(e^{-x}))dx &\lesssim \int_{\delta^\beta}^{\infty} \left(\frac{1-|w_j|^2}{1-e^{-2x}|w_j|^2} e^{-x} \right)^\gamma dx \\ &\lesssim (1-|w_j|)^\gamma \int_{\delta^\beta}^{\infty} \frac{e^{-\gamma x}}{(1-e^{-x}|w_j|)^\gamma} dx \\ &\leq (1-|w_j|)^\gamma |w_j|^{-\gamma} \int_0^1 \frac{(1-u)^{\gamma-1}}{u^\gamma} du \leq C_\gamma \delta^{\gamma/4}. \end{aligned}$$

where we have made the change of variables $u = 1 - e^{-x}|w_j|$. We therefore have

$$\int_D G(\zeta_j, z) \Delta\phi_L(z) \lesssim (\delta^{\gamma/4} + \delta^\beta) L$$

and the claim follows by choosing $\alpha = \min\{\gamma/4, \beta\}$. ■

We are now ready to prove Theorem 6. Since we do not have any control on the dependence of the constants on the bounded set U , we assume that U is the disc D .

Proof of Theorem 6. Suppose that g_L has no zeroes in D . Recall that we use $G(\zeta, z)$ to denote the Green function for D . Applying Jensen's formula to g_L and the Riesz decomposition to the subharmonic function ϕ_L on the disc D we see that

$$\log |g_L(z_0)| - \phi_L(z_0) = \int_{\partial D} (\log |g_L(z)| - \phi_L(z)) + L \int_D G(z_0, z) \Delta\phi(z) dm(z).$$

Choosing $\delta = \int_D G(z_0, z) \Delta\phi(z)/2$ in Lemma 25 shows that outside an exceptional set of probability at most e^{-cL^2} we have

$$\log |g_L(\zeta_0)| - \phi_L(\zeta_0) \geq \delta L.$$

Now Proposition 14 shows that

$$\mathbb{P}[\log |g_L(\zeta_0)| - \phi_L(\zeta_0) \geq \delta L] \leq \mathbb{P}\left[\frac{|g_L(\zeta_0)|}{\mathcal{K}_L(z_0, z_0)^{1/2}} \gtrsim e^{\delta L}\right] \leq \exp\{-Ce^{2\delta L}\} \leq e^{-cL^2}$$

and so

$$\mathbb{P}[n_L(D) = 0] \leq e^{-cL^2},$$

which completes the proof. ■

7. PROOF OF THEOREM 7

We have previously remarked that the upper bound in Theorem 7 is a simple consequence of Theorem 4, we now prove the lower bounds. We will do this by first finding a deterministic function h_L that does not vanish in the hole and then constructing an event that ensures the GAF f_L is 'close' to h_L . Since we can always find a disc $D = D^r(z_0)$ contained in U , and we do not

have any control on the dependence of the constants on U , we will prove the theorem only in the case $U = D$. We begin by constructing the function h_L .

Lemma 29. *There exists an entire function h_L with the following properties:*

- $\|h_L\|_{\mathcal{F}_L^2} = 1$.
- There exists $C_0 > 0$ depending on $\mu(D)$ and the doubling constant such that

$$|h_L(z)|e^{-\phi_L(z)} \geq e^{-C_0 L}$$

for all $z \in D$.

Remark. In the case $\phi(z) = |z|^2$ we may take h_L to be constant. More generally, if

$$\int_{\mathbb{C}} e^{-2\phi_L} \frac{dm}{\rho_L^2} \leq C^L$$

then we can take $h_L = C^{-L}$. In general, however, it may not even be the case that

$$\int_{\mathbb{C}} e^{-2\phi} \frac{dm}{\rho^2}$$

is finite (consider $\phi(z) = (\operatorname{Re} z)^2$).

Proof. Let $\mathcal{K}_\delta(z, w)$ be the reproducing kernel for the space $\mathcal{F}_{\delta\phi}^2$ and consider the normalised reproducing kernel

$$k_\delta(z) = \frac{\mathcal{K}_\delta(z, z_0)}{\mathcal{K}_\delta(z_0, z_0)^{1/2}}.$$

Now since $\rho_{\delta\mu}(z_0) \rightarrow \infty$ as $\delta \rightarrow 0$ Proposition 14 shows that there exists δ_0 and $C > 0$ (depending only on r and the doubling constant) such that

$$(14) \quad |k_\delta(z)|e^{-\delta\phi(z)} \geq C$$

for all $z \in D$ and all $\delta < \delta_0$. Given any L sufficiently large we can find $\delta \in [\delta_0/2, \delta_0]$ and an integer N such that $L = N\delta$. We note that $\rho_{\delta\mu}(z) \simeq \rho_\mu(z)$ for all δ in this range (where the implicit constants depend on δ_0) and so applying Proposition 14 and (7) gives

$$\begin{aligned} \int_{\mathbb{C}} |k_\delta(z)|^{2N} e^{-2\phi_L(z)} \frac{dm(z)}{\rho_L(z)^2} &\lesssim L^{2/\gamma} \int_{\mathbb{C}} (|k_\delta(z)|e^{-\delta\phi(z)})^{2N} \frac{dm(z)}{\rho_{\delta\mu}(z)^2} \\ &\lesssim L^{2/\gamma} \int_{\mathbb{C}} e^{-d_{\delta\phi}^{\mathbb{C}}(z, z_0)} \frac{dm(z)}{\rho_{\delta\mu}(z)^2} \lesssim L^{2/\gamma}. \end{aligned}$$

Hence k_δ^N is an entire function in $\mathcal{F}_{\phi_L}^2$ and we define $h_L = k_\delta^N / \|k_\delta^N\|_{\mathcal{F}_{\phi_L}^2}$. We finally note that (14) implies that for all $z \in D$

$$|h_L(z)|e^{-\phi_L(z)} = (|k_\delta(z)|e^{-\delta\phi(z)})^N / \|k_\delta^N\|_{\mathcal{F}_{\phi_L}^2} \gtrsim C^N L^{1/\gamma} \geq e^{-C_0 L}$$

where C_0 depends on δ_0 and the doubling constant. ■

Proof of the upper bounds in Theorem 7. (a) Recall that $(\tilde{k}_\lambda)_{\lambda \in \Lambda_L}$ is the dual frame associated to the frame $(k_\lambda)_{\lambda \in \Lambda_L}$. Since $h_L \in \mathcal{F}_{\phi_L}^2$ we may write $h_L = \sum_{\lambda \in \Lambda} \langle h_L, \tilde{k}_\lambda \rangle k_\lambda = \sum_{\lambda \in \Lambda} c_\lambda k_\lambda$ where we define $c_\lambda = \langle h_L, \tilde{k}_\lambda \rangle$ (and we ignore the dependence on L to simplify the notation). Note that, for any $z \in D$, we have

$$|f_L(z)|e^{-\phi_L(z)} = \left| h_L(z) + \sum_{\lambda \in \Lambda} (a_\lambda - c_\lambda) \tilde{k}_\lambda(z) \right| e^{-\phi_L(z)} \geq e^{-C_0 L} - \sum_{\lambda \in \Lambda} |a_\lambda - c_\lambda| |\tilde{k}_\lambda(z)| e^{-\phi_L(z)}.$$

We therefore have

$$\mathbb{P}[n_L(D) = 0] \geq \mathbb{P}[\max_{z \in D} \sum_{\lambda \in \Lambda} |a_\lambda - c_\lambda| |\tilde{k}_\lambda(z)| e^{-\phi_L(z)} < e^{-C_0 L}]$$

and we now estimate the probability of this event. First define

$$\alpha = \max\{0, \frac{1}{\delta}(\frac{1}{\epsilon} - \gamma)\}$$

where ϵ, γ and δ are the constants appearing in Proposition 14, (7) and Lemma 11 respectively. Fix a large positive constant C_1 to be specified, write

$$D_L = D^{C_1 L^{\alpha r}}(z_0)$$

and define the event

$$\mathcal{E}_1 = \{|a_\lambda - c_\lambda| \leq L \frac{|\lambda - z_0|}{\rho_L(z_0)} : \lambda \in \Lambda_L \setminus D_L\}.$$

If \mathcal{E}_1 occurs and $z \in D$ then, using an argument identical to that given in the proof of Lemma 12, we see that

$$\begin{aligned} \sum_{\lambda \in \Lambda_L \setminus D_L} |a_\lambda - c_\lambda| |\tilde{k}_\lambda(z)| e^{-\phi_L(z)} &\lesssim L \sum_{\lambda \in \Lambda_L \setminus D_L} \frac{|\lambda - z_0|}{\rho_L(z_0)} e^{-d_L^\epsilon(z, \lambda)} \\ &\lesssim L^{1+1/\gamma} \sum_{\lambda \in \Lambda_L \setminus D_L} \frac{|\lambda - z_0|}{\rho(z_0)} e^{-c' L^{\epsilon \gamma} d_\phi^\epsilon(z_0, \lambda)} \\ &\lesssim L^{1+1/\gamma} \int_{\mathbb{C} \setminus D_L} \frac{|\zeta - z_0|}{\rho(z_0)} e^{-c' L^{\epsilon \gamma} \left(\frac{|\zeta - z_0|}{\rho(z_0)}\right)^{\epsilon \delta}} \frac{dm(z)}{\rho_L(\zeta)^2} \\ &\lesssim L^{\beta_0} \int_{c' C_1^{\delta \epsilon} L^{\alpha'}}^{+\infty} e^{-t} t^{\beta_1} dt \\ &\leq \frac{1}{2} e^{-C_0 L} \end{aligned}$$

for C_1 sufficiently large, where $\alpha' = \max\{1, \epsilon \gamma\}$, and β_0 and $\beta_1 > 0$ are some exponents that depend on the doubling constant.

We define the event

$$\mathcal{E}_2 = \{|a_\lambda - c_\lambda| \leq \frac{e^{-C_0 L}}{C_2 \sqrt{\#\Lambda_L \cap D_L}} : \lambda \in \Lambda_L \cap D_L\},$$

where C_2 is a positive constant to be chosen. Note that for all $z \in \mathbb{C}$, \mathcal{E}_2 implies that by choosing C_2 sufficiently large

$$\begin{aligned} \sum_{\lambda \in \Lambda_L \cap D_L} |a_\lambda - c_\lambda| |k_\lambda(z)| e^{-\phi_L(z)} &\leq \left(\sum_{\lambda \in \Lambda_L \cap D_L} |a_\lambda - c_\lambda|^2 \right)^{1/2} \left(\sum_{\lambda \in \Lambda_L \cap D_L} |k_\lambda(z)|^2 \right)^{1/2} e^{-\phi_L(z)} \\ &\leq \frac{e^{-C_0 L}}{C_2} K_L(z, z)^{1/2} e^{-\phi_L(z)} < \frac{1}{2} e^{-C_0 L}. \end{aligned}$$

Hence

$$\mathbb{P}[n_L(D) = 0] \geq \mathbb{P}[\mathcal{E}_1] \mathbb{P}[\mathcal{E}_2]$$

Recalling the definition of the coefficients c_λ we note that

$$\sum_{\lambda \in \Lambda_L} |c_\lambda|^2 \simeq \|h_L\|_{\mathcal{F}_{\phi_L}^2}^2 = 1$$

and so the coefficients c_λ are bounded. This means that

$$\mathbb{P}\left[|a_\lambda - c_\lambda| \leq L \frac{|\lambda - z_0|}{\rho_L(z_0)}\right] \geq \mathbb{P}\left[|a_\lambda| \leq L \frac{|\lambda - z_0|}{2\rho_L(z_0)}\right]$$

when $\lambda \in \Lambda_L \setminus D_L$ and L is large. We may estimate $\mathbb{P}[\mathcal{E}_1]$ similarly to (13) in the proof of Lemma 20. This yields $\mathbb{P}[\mathcal{E}_1] \geq 1/2$ for large L .

Finally since $\#\Lambda_L \cap D_L \simeq \Delta_{\phi_L}(D_L) \lesssim L^{1+\alpha/\gamma}$ we have

$$\mathbb{P}[\mathcal{E}_2] = \prod_{\lambda \in \Lambda_L \cap D_L} \mathbb{P}\left[|a_\lambda - c_\lambda| \leq \frac{e^{-C_0 L}}{C_2 \sqrt{\#\Lambda_L \cap D_L}}\right] \geq \left(C \frac{e^{-2C_0 L}}{\#\Lambda_L \cap D_L}\right)^{\#\Lambda_L \cap D_L} \geq e^{-cL^{2+\alpha/\gamma}}$$

for some positive constants C and c . Considering the two possible values of α completes the proof of the lower bounds in Theorem 7.

(b) We assume that the reproducing kernel \mathcal{K}_L satisfies the estimate (10). We will use the same notation as before. Let C_3 and C_4 be constants to be chosen and define the following events

$$\begin{aligned} \mathcal{A}_1 &= \{|a_\lambda - c_\lambda| \leq L \frac{|\lambda - z_0|}{\rho_L(z_0)} : \lambda \in \Lambda_L \setminus D^{C_3 r}(z_0)\} \\ \mathcal{A}_2 &= \{|c_\lambda - a_\lambda| \leq \frac{e^{-C_0 L}}{C_4 \sqrt{\#\Lambda_L \cap D^{C_3 r}(z_0)}} : \lambda \in \Lambda_L \cap D^{C_3 r}(z_0)\}. \end{aligned}$$

We have already seen that the event \mathcal{A}_1 implies that

$$\left| \sum_{\lambda \in \Lambda_L \setminus D_L} (a_\lambda - c_\lambda) \tilde{k}_\lambda(z) \right| e^{-\phi_L(z)} \leq \frac{1}{2} e^{-C_0 L}$$

for $z \in D$. We write $\tilde{\Lambda}_L = \Lambda_L \cap (D_L \setminus D^{C_3 r}(z_0))$. Note that \mathcal{A}_1 and (10) imply that, for $z \in D$,

$$\begin{aligned} & \sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda - c_\lambda| |k_\lambda(z)| e^{-\phi_L(z)} \\ & \leq \left(\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda - c_\lambda|^2 \right)^{1/2} \left(\sum_{\lambda \in \tilde{\Lambda}_L} |k_\lambda(z)|^2 \right)^{1/2} e^{-\phi_L(z)} \\ & \lesssim L^{1+\alpha+1/\gamma} \sqrt{\#\tilde{\Lambda}_L} \left(\int_{\mathbb{C} \setminus D^{C_3 r}(z_0)} |\mathcal{K}_L(z, \zeta)| e^{-2\phi_L(\zeta)} \frac{dm(\zeta)}{\rho_L(\zeta)^2} \right)^{1/2} e^{-\phi_L(z)} \\ & < \frac{1}{4} e^{-C_0 L} \end{aligned}$$

for an appropriately large choice of C_3 and for all large L . By an identical computation to before we see that \mathcal{A}_2 implies that for all $z \in \mathbb{C}$

$$\sum_{\lambda \in \Lambda_L \cap D^{C_3 r}(z_0)} |a_\lambda - c_\lambda| |k_\lambda(z)| e^{-\phi_L(z)} < \frac{1}{4} e^{-C_0 L}$$

by choosing C_4 sufficiently large. It remains only to estimate the probabilities of the events \mathcal{A}_1 and \mathcal{A}_2 , which are again identical to the previous computation. This completes the proof. \blacksquare

APPENDIX: THE CASE $|z|^\alpha/2$

We consider the space \mathcal{F}_L^2 when $\phi(z) = |z|^\alpha/2$ and we first note that for $|z| \leq 1$

$$\rho(z) \simeq 1$$

and that

$$\rho(z) \simeq |z|^{1-\alpha/2}$$

otherwise. We begin by showing that the set $(\frac{(L^{1/\alpha} z)^n}{c_{\alpha n}})_{n=0}^\infty$ is an orthonormal basis for some choice of $c_{\alpha n} \simeq \Gamma(\frac{2}{\alpha}n + 1)^{1/2}$. It is clear that the functions z^n are orthogonal because ϕ_L (and therefore ρ_L) are radial, and so we need only compute the appropriate normalising constants

$$\|z^n\|_{\mathcal{F}_L^2}^2 = \int_{\mathbb{C}} |z|^{2n} e^{-L|z|^\alpha} \frac{dm(z)}{\rho_L(z)^2}.$$

Now it is easy to see that for $|z| \leq \rho_L(0)$

$$\rho_L(z) \simeq \rho_L(0) \simeq L^{-1/\alpha}$$

and that

$$\rho_L(z) \simeq L^{-1/2} |z|^{1-\alpha/2}$$

otherwise. Hence, using the fact that $L\rho_L(0)^\alpha \simeq 1$, we have

$$\begin{aligned}
\|z^n\|_{\mathcal{F}_L^2}^2 &= \int_{\mathbb{C}} |z|^{2n} e^{-L|z|^\alpha} \frac{dm(z)}{\rho_L(z)^2} \\
&\simeq L^{2/\alpha} \int_{D^L(0)} |z|^{2n} e^{-L|z|^\alpha} dm(z) + L \int_{\mathbb{C} \setminus D^L(0)} |z|^{2n} e^{-L|z|^\alpha} |z|^{\alpha-2} dm(z) \\
&\simeq L^{-2n/\alpha} \left(\int_0^{L\rho_L(0)^\alpha} u^{1+(2n+2)/\alpha} e^{-u} du + \int_{L\rho_L(0)^\alpha}^\infty u^{2n/\alpha} e^{-u} du \right) \\
&\simeq L^{-2n/\alpha} \Gamma\left(\frac{2}{\alpha}n + 1\right)^{1/2}.
\end{aligned}$$

It follows that, for some coefficients $c_{\alpha n} \simeq \Gamma(\frac{2}{\alpha}n + 1)^{1/2}$, the set $(\frac{(L^{1/\alpha}z)^n}{c_{\alpha n}})_{n=0}^\infty$ is an orthonormal basis for \mathcal{F}_L^2 and the reproducing kernel for this space is then given by

$$\mathcal{K}_L(z, w) = \sum_{n=0}^\infty \frac{(L^{2/\alpha} z \bar{w})^n}{c_{\alpha n}^2}.$$

We recall that for positive a the Mittag-Leffler function

$$E_{a,1}(\zeta) = \sum_{n=0}^\infty \frac{\zeta^n}{\Gamma(an + 1)}$$

is an entire function of order $1/a$ satisfying

$$E_{a,1}(x) \lesssim e^{x^{1/a}}$$

for all real positive x .

We now show that \mathcal{K}_L has fast L^2 off-diagonal decay, that is, given $C, r > 0$ there exists $R > 0$ (independent of L) such that

$$\sup_{z \in D^r(z_0)} e^{-L|z|^\alpha} \int_{\mathbb{C} \setminus D^{2R}(z_0)} |\mathcal{K}_L(z, w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \leq e^{-CL}$$

for all $z_0 \in \mathbb{C}$ and L sufficiently large (we have replaced R by $2R$ to simplify the notation in what follows). Choosing R sufficiently large we have

$$\int_{\mathbb{C} \setminus D^{2R}(z_0)} |\mathcal{K}_L(z, w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \leq \int_{\mathbb{C} \setminus D(0,R)} |\mathcal{K}_L(z, w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2}$$

and we note again that ϕ_L and ρ_L are radial. Thus, for any positive integers n and m ,

$$\begin{aligned} \int_{\mathbb{C} \setminus D(0,R)} w^n \bar{w}^m e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} &\simeq \delta_{nm} \int_{\mathbb{C} \setminus D(0,R)} |w|^{2n} e^{-L|w|^\alpha} L|w|^{\alpha-2} dm(w) \\ &= 2\pi \delta_{nm} \int_R^\infty r^{2n} e^{-Lr^\alpha} Lr^{\alpha-1} dr \\ &= \frac{2\pi}{\alpha} \delta_{nm} L^{-2n/\alpha} \int_{LR^\alpha}^\infty u^{2n/\alpha} e^{-u} du \\ &= \frac{2\pi}{\alpha} \delta_{nm} L^{-2n/\alpha} \Gamma\left(\frac{2}{\alpha}n + 1, LR^\alpha\right) \end{aligned}$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ denotes the incomplete Gamma function. Now, recalling the expression for the kernel \mathcal{K}_L we see that

$$|\mathcal{K}_L(z, w)|^2 = \sum_{m,n=0}^\infty \frac{L^{2(m+n)/\alpha}}{c_{\alpha m} c_{\alpha n}} z^m \bar{z}^n \bar{w}^m w^n$$

and so

$$\begin{aligned} \int_{\mathbb{C} \setminus D(0,R)} |\mathcal{K}_L(z, w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} &= \sum_{m,n=0}^\infty \frac{L^{2(m+n)/\alpha}}{c_{\alpha m} c_{\alpha n}} z^m \bar{z}^n \int_{\mathbb{C} \setminus D(0,R)} w^n \bar{w}^m e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \\ &\simeq \sum_{n=0}^\infty \frac{L^{4n/\alpha}}{\Gamma(\frac{2}{\alpha}n + 1)^2} |z|^{2n} L^{-2n/\alpha} \Gamma\left(\frac{2}{\alpha}n + 1, LR^\alpha\right) \\ &= \sum_{n=0}^\infty \frac{(L^{2/\alpha} |z|^2)^n}{\Gamma(\frac{2}{\alpha}n + 1)} \frac{\Gamma(\frac{2}{\alpha}n + 1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n + 1)}. \end{aligned}$$

We split this sum in two parts. Choose $N = \lceil \frac{\alpha}{4} LR^\alpha \rceil$ and note that for $n \leq N$ we have, by standard estimates for the incomplete Gamma function,

$$\Gamma\left(\frac{2}{\alpha}n + 1, LR^\alpha\right) \simeq (LR^\alpha)^{2n/\alpha} e^{-LR^\alpha}$$

as $R \rightarrow \infty$. Now Stirling's approximation shows that

$$\begin{aligned} \frac{(LR^\alpha)^{\frac{2}{\alpha}n}}{\Gamma(\frac{2}{\alpha}n + 1)} &\leq \frac{(LR^\alpha)^{\frac{2}{\alpha}N}}{\Gamma(\frac{2}{\alpha}N + 1)} \\ &\simeq \left(\frac{4}{\alpha}N\right)^{\frac{2}{\alpha}N} \left(\frac{2}{\alpha}N + 1\right)^{1/2} \left(\frac{e}{\frac{2}{\alpha}N + 1}\right)^{\frac{2}{\alpha}N+1} \\ &\lesssim \left(\frac{\frac{4}{\alpha}N}{\frac{2}{\alpha}N + 1}\right)^{\frac{2}{\alpha}N} e^{\frac{2}{\alpha}N} \\ &\lesssim 2^{\frac{2}{\alpha}N} e^{\frac{2}{\alpha}N} \\ &= e^{LR^\alpha(1+\log 2)/2} \end{aligned}$$

and so

$$\frac{\Gamma(\frac{2}{\alpha}n + 1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n + 1)} \lesssim e^{-cLR^\alpha}.$$

It follows that

$$\begin{aligned} \sum_{n=0}^N \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n + 1)} \frac{\Gamma(\frac{2}{\alpha}n + 1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n + 1)} &\lesssim e^{-cLR^\alpha} \sum_{n=0}^{\infty} \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n + 1)} \\ &= e^{-cLR^\alpha} E_{\frac{2}{\alpha},1}(L^{2/\alpha}|z|^2) \\ &\lesssim e^{-cLR^\alpha} e^{L|z|^\alpha}. \end{aligned}$$

To deal with the remaining terms we first note that

$$\frac{\Gamma(\frac{2}{\alpha}n + 1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n + 1)} \leq 1$$

for all n . We now choose R so large that

$$L^{2/\alpha}|z|^2 < e^{-4/\alpha} L^{2/\alpha} R^2 < e^{-4/\alpha} \left(\frac{2}{\alpha} N + 1 \right)^{2/\alpha}$$

for $z \in D^r(z_0)$. Note that another application of Stirling's approximation yields, for any $n > N$,

$$\Gamma\left(\frac{2}{\alpha}n + 1\right) \gtrsim \Gamma\left(\frac{2}{\alpha}N + 1\right) \left(\frac{2}{\alpha}N + 1\right)^{2(n-N)/\alpha}.$$

We conclude that for $z \in D^r(z_0)$ and R sufficiently large we have

$$\begin{aligned} \sum_{n>N} \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n + 1)} \frac{\Gamma(\frac{2}{\alpha}n + 1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n + 1)} &\lesssim \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N + 1)} \sum_{n=0}^{\infty} \frac{(L^{2/\alpha}|z|^2)^n}{(\frac{2}{\alpha}N + 1)^{2n/\alpha}} \\ &= \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N + 1)} \left(1 - \frac{L^{2/\alpha}|z|^2}{(\frac{2}{\alpha}N + 1)^{2/\alpha}}\right)^{-1} \\ &\simeq \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N + 1)}. \end{aligned}$$

A final appeal to Stirling's approximation yields

$$\begin{aligned} \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N + 1)} &\simeq (L^{2/\alpha}|z|^2)^N \left(\frac{2}{\alpha}N + 1\right)^{1/2} \left(\frac{e}{\frac{2}{\alpha}N + 1}\right)^{\frac{2}{\alpha}N+1} \\ &\simeq \left(\frac{L^{2/\alpha}|z|^2}{(\frac{2}{\alpha}N + 1)^{2/\alpha}}\right)^N \left(\frac{2}{\alpha}N + 1\right)^{-1/2} e^{2N/\alpha} \\ &\lesssim e^{-4N/\alpha} e^{2N/\alpha} \\ &= e^{-LR^\alpha/2}. \end{aligned}$$

Retracing our footsteps we see that we have shown that

$$\int_{\mathbb{C} \setminus D^{2R}(z_0)} |\mathcal{K}_L(z, w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \lesssim e^{-cLR^\alpha} (1 + e^{L|z|^\alpha})$$

for all $z \in D^r(z_0)$ and R sufficiently large. Hence

$$\sup_{z \in D^r(z_0)} e^{-L|z|^\alpha} \int_{\mathbb{C} \setminus D^{2R}(z_0)} |\mathcal{K}_L(z, w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \leq e^{-cL}$$

for an appropriately large R , as claimed.

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